Decoupled, unconditionally stable, higher order discretizations for MHD flow simulation

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Abstract

We propose, analyze, and test a new MHD discretization which decouples the system into two Oseen problems at each timestep yet maintains unconditional stability with respect to the time step size, is optimally accurate in space, and behaves like second order in time in practice. The proposed method chooses a parameter $\theta \in [0,1]$, dependent on the viscosity ν and magnetic diffusivity ν_m, so that the explicit treatment of certain viscous terms does not cause instabilities, and gives temporal accuracy $O(\Delta t^2 + (1-\theta)|\nu - \nu_m|\Delta t)$. In practice, ν and ν_m are small, and so the method behaves like second order. When $\theta = 1$, the method reduces to a linearized BDF2 method, but it has been proven by Li and Trenchea that such a method is stable only in the uncommon case of $\frac{1}{2} < \frac{\nu}{\nu_m} < 2$. For the proposed method, stability and convergence are rigorously proven for appropriately chosen θ, and several numerical tests are provided that confirm the theory and show the method provides excellent accuracy in cases where usual BDF2 is unstable.

1 Introduction

Recent advances in algorithms, large scale computing, and understanding of fluid flow phenomena has made it possible to consider the simulation of multiphysics flow problems which couple the Navier-Stokes equations (NSE) to conservation laws and constitutive equations for additional physical phenomena. Our interest herein is with magnetohydrodynamic (MHD) flow, which is important in various applications including astrophysics and geophysics [21, 28, 15, 12, 4, 7], liquid metal cooling in nuclear reactors [3, 18, 30], and process metallurgy [10]. The MHD system of equations is created by nonlinearly coupling the NSE to Maxwell's equation for magnetic fields, and is given in a convex domain Ω by [24, 6, 11]

$$u_t + (u\cdot\nabla)u - s(B\cdot\nabla)B - \nu\Delta u + \nabla p = f, \tag{1.1}$$

$$\nabla\cdot u = 0, \tag{1.2}$$

$$B_t + (u\cdot\nabla)B - (B\cdot\nabla)u - \nu_m\Delta B + \nabla\lambda = \nabla\times g, \tag{1.3}$$

$$\nabla\cdot B = 0. \tag{1.4}$$

Here, Ω is domain of the fluid, u is velocity, p is a modified pressure, f is body force, $\nabla\times g$ is a forcing on the magnetic field B, s is the coupling number, T is the time period, ν is the kinematic

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viscosity and ν_m is the magnetic diffusivity. The conservation of linear momentum is enforced by (1.1) and the conservation of mass by (1.2). Equation (1.3) represents the induction equation for the magnetic field B, and is accompanied by the solenoidal constraint on the magnetic induction as (1.4). Equation (1.4) ensures that there are no magnetic monopoles, which are hypothetical elementary particles with an isolated magnetic north or south pole. The modified pressure p is related to the fluid pressure, p_f, via $p = p_f/\rho + B \cdot B/2$, where density is denoted by ρ. The magnetic diffusivity ν_m is defined by $\nu_m := Re_m^{-1} = 1/(\mu\sigma)$, where μ is the magnetic permeability of free space and σ is the electric conductivity of the fluid. An important property that determines the behavior of the MHD equation is the ratio between the viscous and magnetic diffusion rates, the magnetic Prandtl number $Pr_m = Re_m/Re = \nu/\nu_m$. This ratio is crucial for the stability of our scheme, as it is used to determine a key parameter. The artificial magnetic pressure λ is a Lagrange multiplier, introduced in the induction equation to enforce the divergence free constraint on the magnetic induction equation within a variational context. In the continuous case, the magnetic pressure vanishes.

Since the nonlinear system (1.1)-(1.4) typically requires a very large number of degrees of freedom (dof) to resolve numerically, it is critical to split the system into smaller, more easily solvable pieces to avoid having to solve very large, coupled nonsymmetric linear systems. Even for the NSE alone, the development of fast, robust, linear solvers is very much an open research problem, although some excellent strides forward have been recently made [5, 13, 23]. Thus solving the coupled block linear systems that arise at each time step of an MHD simulation will generally be computationally infeasible.

A breakthrough for efficient MHD algorithms was recently made by C. Trenchea in [31], where he showed that if the MHD system is rewritten in Elsässer variables [14] instead of primitive variables, then the MHD system can be decoupled in an unconditionally stable way into two Oseen problems at every time step. This idea was further explored in [26], and shown to work very well. To describe the method, we first recall the Elsässer variable reformulation of MHD: define $v =: u + \sqrt{s}B$, $w =: u - \sqrt{s}B$, $f_1 := f + \sqrt{s}(\nabla \times g)$, $f_2 := f - \sqrt{s}(\nabla \times g)$, $q := p + \sqrt{s}\lambda$ and $r := p - \sqrt{s}\lambda$. This changes (1.1)-(1.4) to

$$v_t + w \cdot \nabla v + \nabla q - \frac{\nu + \nu_m}{2}\Delta v - \frac{\nu - \nu_m}{2}\Delta w = f_1, \tag{1.5}$$

$$\nabla \cdot v = 0, \tag{1.6}$$

$$w_t + v \cdot \nabla w + \nabla r - \frac{\nu + \nu_m}{2}\Delta w - \frac{\nu - \nu_m}{2}\Delta v = f_2, \tag{1.7}$$

$$\nabla \cdot w = 0. \tag{1.8}$$

The above system can be easily transformed to the case of $B = B_0 + b$, where B_0 is a known uniform background magnetic field and b is fluctuations in it. For simplicity of analysis, however, we will assume $B_0 = 0$, since adding this term would not change the main ideas or results. We note also that certain physical phenomena for MHD turbulence can be more easily described using the Elsässer formulation [9], and that the velocity u and magnetic field B are easily recoverable from simulations using Elsässer variables.

The decoupled and unconditionally stable first order timestepping method of Trenchea has the form

First order decoupled method of Trenchea [31]:

$$\begin{aligned}
\frac{1}{\Delta t}(v^{n+1}-v^n)+w^n\cdot\nabla v^{n+1}+\nabla q^{n+1}-\frac{\nu+\nu_m}{2}\Delta v^{n+1}-\frac{\nu-\nu_m}{2}\Delta w^n &= f_1^{n+1},\\
\nabla\cdot v^{n+1} &= 0,\\
\frac{1}{\Delta t}(w^{n+1}-w^n)+v^n\cdot\nabla w^{n+1}+\nabla r^{n+1}-\frac{\nu+\nu_m}{2}\Delta w^{n+1}-\frac{\nu-\nu_m}{2}\Delta v^n &= f_2^{n+1},\\
\nabla\cdot w^{n+1} &= 0.
\end{aligned}$$

Although a successful breakthrough idea, a drawback to the scheme is that it is limited to first order temporal accuracy. In a followup work, Li and Trenchea studied the following second order extension:

Second order decoupled method of Li and Trenchea [25]:

$$\begin{aligned}
\frac{1}{2\Delta t}(3v^{n+1}-4v^n+v^{n-1})+(2w^n-w^{n-1})\cdot\nabla v^{n+1}+\nabla q^{n+1}&\\
-\frac{\nu+\nu_m}{2}\Delta v^{n+1}-\frac{\nu-\nu_m}{2}\Delta(2w^n-w^{n-1}) &= f_1^{n+1},\\
\nabla\cdot v^{n+1} &= 0,\\
\frac{1}{2\Delta t}(3w^{n+1}-4w^n+w^{n-1})+(2v^n-v^{n-1})\cdot\nabla w^{n+1}+\nabla r^{n+1}&\\
-\frac{\nu+\nu_m}{2}\Delta w^{n+1}-\frac{\nu-\nu_m}{2}\Delta(2v^n-v^{n-1}) &= f_2^{n+1},\\
\nabla\cdot w^{n+1} &= 0,
\end{aligned}$$

and found it was unconditionally stable only under the restriction $\frac{1}{2} < Pr_m = \frac{v}{v_m} < 2$. In [1] this bound was shown to be sharp, and thus there is a serious restriction on its applicability in practice for many problems. For example current estimates suggest $Pr_m \sim 10^{-5}$ in the Earth's core ($Re \sim 10^8$, $Re_m \sim 10^3$, see [22, 27]). We prove herein (in the appendix) that the above second order method can be stable without the restriction on $Pr_m = \frac{\nu}{\nu_m}$, if a timestep restriction of $\Delta t < O(h^2)$ is satisfied. Note that this condition is also often not practical.

The purpose of this paper is to propose and study a decoupled, unconditionally stable and higher order accurate scheme that has no restriction on ν and ν_m. By careful consideration of the analysis in [25, 31], we identify the 'problem terms' that lead to the restriction are the $(\nu-\nu_m)$ terms. In the first order case, these can be handled, but in the second order case, a restriction on the data becomes necessary. Thus, we propose a method that treats the $(\nu-\nu_m)$ terms as a linear combination (i.e. a θ-method) of the first and second order schemes above, which takes the form

Proposed decoupled θ-method:

$$\begin{aligned}
\frac{1}{2\Delta t}(3v^{n+1}-4v^n+v^{n-1})+(2w^n-w^{n-1})\cdot\nabla v^{n+1}+\nabla q^{n+1}&\\
-\frac{\nu+\nu_m}{2}\Delta v^{n+1}-\theta\frac{\nu-\nu_m}{2}\Delta(2w^n-w^{n-1})-(1-\theta)\frac{\nu-\nu_m}{2}\Delta w^n &= f_1^{n+1},\\
\nabla\cdot v^{n+1} &= 0,\\
\frac{1}{2\Delta t}(3w^{n+1}-4w^n+w^{n-1})+(2v^n-v^{n-1})\cdot\nabla w^{n+1}+\nabla r^{n+1}&\\
-\frac{\nu+\nu_m}{2}\Delta w^{n+1}-\theta\frac{\nu-\nu_m}{2}\Delta(2v^n-v^{n-1})-(1-\theta)\frac{\nu-\nu_m}{2}\Delta v^n &= f_2^{n+1},\\
\nabla\cdot w^{n+1} &= 0,
\end{aligned}$$

For this method, we prove unconditional stability of the method for any ν and ν_m, provided θ is chosen to satisfy $\frac{\theta}{1+\theta}<\frac{\nu}{\nu_m}<\frac{1+\theta}{\theta}$, $0\le\theta\le 1$. This can be achieved for any $\frac{\nu}{\nu_m}$, because the bounds tend towards negative and positive infinity for θ going to zero. We also prove this scheme has temporal accuracy $O(\Delta t^2+(1-\theta)|\nu-\nu_m|\Delta t)$. Even though the method is not second order unless $\theta=1$ (the case where the BDF2 scheme is stable), in practice ν and ν_m are typically small, and thus the method will typically behave like a second order method. To return to the example of Earth's core, there $|\nu-\nu_m|$ is in the order of 10^{-3}. We also note that the two decoupled Oseen problems can be solved independently, allowing for a parallel solution approach if desired.

We study the new decoupled θ-method in a fully discrete setting, using a finite element spatial discretization. After providing some necessary notation and mathematical preliminaries in Section 2, we prove the proposed scheme is unconditionally stable (with correct choice of θ), well-posed, optimally accurate in space, and with temporal accuracy $O(\Delta t^2+(1-\theta)|\nu-\nu_m|\Delta t)$, *without any restrictions on ν and ν_m* (see Section 3). The proposed method is the only unconditionally stable, decoupled method for MHD with general ν and ν_m that is better than first order accurate in time, and thus could represent a potentially significant step forward for MHD flow simulations. In Section 4 we perform several numerical experiments that both validate the theory and show the method is very effective on some benchmark problems where the full second order method of [25] is unstable.

2 Notation and Preliminaries

Throughout this paper, we assume that $\Omega\subset\mathbb{R}^d, d\in 2,3$, is a convex polygonal or polyhedral domain with boundary $\partial\Omega$. We denote the usual $L^2(\Omega)$ norm and its inner product by $\|\cdot\|$ and $(\cdot,\cdot)$ respectively. All other norms will be clearly labeled.

For X being a normed function space in Ω, $L^p(0,t;X)$ is the space of all functions defined on $(0,t)\times\Omega$ for which the norm

$$\|u\|_{L^p(0,t;X)}=\left(\int_0^t\|u\|_X^p\,\mathrm{d}x\right)^{1/p},p\in[1,\infty)$$

is finite. For $p=\infty$, the usual modification is used in the definition of this space.

The natural function spaces for our problem are

$$X:=H_0^1(\Omega)^d=\{v\in(L^2(\Omega))^d:\nabla v\in L^2(\Omega)^{d\times d},v=0\text{ on }\partial\Omega\},$$

$$Q := L^2_0(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q \quad \mathrm{d}x = 0\}.$$

The results of this paper also hold in the periodic setting.

The Poincaré-Friedrichs' inequality will be used frequently throughout our analysis: For $v \in X$,

$$\|v\| \leq C\|\nabla v\|, \;\; C = C(\Omega).$$

The space of divergence free functions in X is given by

$$V := \{v \in X : (\nabla \cdot v, q) = 0, \forall q \in Q\}.$$

For f an element in the dual space of X, its norm is defined by

$$\|f\|_{-1} := \sup_{v\in X} \frac{\|(f, v)\|}{\|\nabla v\|}.$$

We define the trilinear form $b^* : X \times X \times X \to \mathbb{R}$ by

$$b^*(u, v, w) := \frac{1}{2}((u \cdot \nabla v, w) - (u \cdot \nabla w, v)).$$

Note that $b^*(u, v, w)$ is skew symmetric, $b^*(u, v, v) = 0$, and if $\|\nabla \cdot u\| = 0$, then $(u \cdot \nabla v, w) = b^*(u, v, w)$. Also, $b^*(u, v, w)$ satisfies the following bound [16]:

$$|b^*(u, v, w)| \leq C(\Omega)\|\nabla u\|\|\nabla v\|\|\nabla w\|, \;\; \text{for any} \;\; u, v, w \in X. \tag{2.1}$$

2.1 Discrete setting

We will assume conforming finite element spaces $X_h \subset X$ and $Q_h \subset Q$ which are LBB stable in the sense of

$$\inf_{q_h\in Q_h} \sup_{v_h\in X_h} \frac{(q_h, \nabla \cdot v_h)}{\|q_h\|\|\nabla v_h\|} \geq \beta > 0, \tag{2.2}$$

where β is independent of h.

For simplicity of analysis, we will further assume that Scott-Vogelius elements are used, i.e., $(X_h, Q_h) = ((P_k)^d, P^{disc}_{k-1})$ with appropriate macro-element structures so that LBB holds [2, 33, 29, 34]. The analysis can easily be extended to any LBB stable pair, e.g. Taylor-Hood elements, with similar analytical results. However, strong enforcement of the $\nabla \cdot B = 0$ constraint is well known to be critical in MHD simulations and thus it seems reasonable to assume the elements used strongly enforce this constraint.

The space of discretely divergence free functions is defined as

$$V_h := \{v_h \in X_h : (\nabla \cdot v_h, q_h) = 0, \;\; \forall \;\; q_h \in Q_h\}.$$

We will formulate our equations in V_h formulation, and due to the LBB condition, this will be equivalent to the (X_h, Q_h) formulation. As is commonly done, we analyze with the V_h formulation and compute with the (X_h, Q_h) form.

With the use of Scott-Vogelius finite element pairs, V_h is conforming to V, i.e., $V_h \subset V$ and the functions in V_h are divergence-free point wise in the L^2 sense:

$$V_h = \{v_h \in X_h, \ \|\nabla \cdot v_h\| = 0\}.$$

We have the following approximation properties in (X_h, Q_h): [8]

$$\inf_{v_h \in X_h} \|u - v_h\| \le Ch^{k+1}|u|_{k+1}, \ \ u \in H^{k+1}(\Omega), \tag{2.3}$$

$$\inf_{v_h \in X_h} \|\nabla(u - v_h)\| \le Ch^{k}|u|_{k+1}, \quad u \in H^{k+1}(\Omega), \tag{2.4}$$

$$\inf_{q_h \in Q_h} \|p - q_h\| \le Ch^{k}|p|_{k}, \qquad p \in H^{k}(\Omega), \tag{2.5}$$

where $|\cdot|_r$ denotes the H^r seminorm.

We will assume the mesh is sufficiently regular for the inverse inequality to hold, and with this and the LBB assumption, we have approximation properties

$$\|\nabla(u - P_{L^2}^{V_h}(u))\| \le Ch^{k}|u|_{k+1}, \ \ u \in H^{k+1}(\Omega), \tag{2.6}$$

$$\inf_{v_h \in V_h} \|\nabla(u - v_h)\| \le Ch^{k}|u|_{k+1}, \ \ u \in H^{k+1}(\Omega), \tag{2.7}$$

where $P_{L^2}^{V_h}(u)$ is the L^2 projection of u into V_h.

The following lemma for the discrete Gronwall inequality was given in [20].

Lemma 2.1. *Let Δt, H, a_n, b_n, c_n, d_n be non-negative numbers for $n = 1, \cdots, M$ such that*

$$a_M + \Delta t \sum_{n=1}^{M} b_n \le \Delta t \sum_{n=1}^{M-1} d_n a_n + \Delta t \sum_{n=1}^{M} c_n + H \quad \textit{for } M \in \mathbb{N},$$

then for all $\Delta t > 0$,

$$a_M + \Delta t \sum_{n=1}^{M} b_n \le exp\left(\Delta t \sum_{n=1}^{M-1} d_n\right)\left(\Delta t \sum_{n=1}^{M} c_n + H\right) \quad \textit{for } M \in \mathbb{N}.$$

3 An efficient and stable θ-scheme for MHD

We now present and analyze an efficient decoupled scheme for MHD. After defining the scheme, we analyze its stability and convergence. The scheme is a generalization of a linearized BDF2 scheme applied to the Elsässer MHD system, and differs in the treatment of the $\frac{\nu - \nu_m}{2}$ terms. As is common with BDF2 schemes, we need two initial conditions; if only one is known, then a linearized backward Euler method (i.e. the first order method of Trenchea [31]) can be used on the first step without affecting stability or accuracy.

Algorithm 3.1. *Given ν and ν_m, choose θ sufficiently small so that $\frac{\theta}{1+\theta} < \frac{\nu}{\nu_m} < \frac{1+\theta}{\theta}$, $0 \le \theta \le 1$. Let $f_1, f_2 \in L^\infty(0, T; H^{-1}(\Omega)^d)$, initial conditions $v^0, w^0, v^1, w^1 \in V_h$, time step $\Delta t > 0$ and end time $T > 0$ be given. Set $M = T/\Delta t$ and for $n = 1, \cdots, M-1$, compute:*

Find $v_h^{n+1} \in V_h$ satisfying, for all $\chi_h \in V_h$,:

$$\left(\frac{3v_h^{n+1} - 4v_h^n + v_h^{n-1}}{2\Delta t}, \chi_h\right) + b^*(2w_h^n - w_h^{n-1}, v_h^{n+1}, \chi_h) + \frac{\nu + \nu_m}{2}\left(\nabla v_h^{n+1}, \nabla \chi_h\right) + \frac{\nu - \nu_m}{2}\left((1-\theta)\nabla w_h^n + \theta\nabla(2w_h^n - w_h^{n-1}), \nabla\chi_h\right) = (f_1(t^{n+1}), \chi_h), \quad (3.1)$$

Find $w_h^{n+1} \in V_h$ satisfying, for all $l_h \in V_h$:

$$\left(\frac{3w_h^{n+1} - 4w_h^n + w_h^{n-1}}{2\Delta t}, l_h\right) + b^*(2v_h^n - v_h^{n-1}, w_h^{n+1}, l_h) + \frac{\nu + \nu_m}{2}\left(\nabla w_h^{n+1}, \nabla l_h\right) + \frac{\nu - \nu_m}{2}\left((1-\theta)\nabla v_h^n + \theta\nabla(2v_h^n - v_h^{n-1}), \nabla l_h\right) = (f_2(t^{n+1}), l_h). \quad (3.2)$$

Remark 3.1. *The key to the efficiency of the scheme is that the equations (3.1) and (3.2) are decoupled; in fact, they could be solved simultaneously if the computational resources are available. We prove below the scheme maintains stability despite this decoupling, provided θ is chosen so that $\frac{\theta}{1+\theta} < \frac{\nu}{\nu_m} < \frac{1+\theta}{\theta}$, $0 \le \theta \le 1$.*

Remark 3.2. *Note that when $\theta = 1$, the above scheme reduces to a linearized BDF2 scheme studied by Li and Trenchea in [25]. However, in [25] it was proven that this case is unconditionally stable when $\frac{1}{2} < \frac{\nu}{\nu_m} < 2$, and it was later verified in [1] that this bound is sharp. This lack of stability is the motivation for the θ-scheme we propose above, since one cannot expect such a restriction on ν and ν_m in general.*

3.1 Stability analysis

We now prove unconditional stability and well-posedness for the Algorithm 3.1. To simplify notation, denote $\alpha := \nu + \nu_m - |\nu - \nu_m|(1 + 2\theta)$, and note that by the choice of θ, it holds that $\alpha > 0$.

Lemma 3.1. *Solutions to Algorithm (3.1) are unconditionally stable: for any $\Delta t > 0$,*

$$\|v_h^M\|^2 + \|2v_h^M - v_h^{M-1}\|^2 + \|w_h^M\|^2 + \|2w_h^M - w_h^{M-1}\|^2 + \alpha\Delta t\sum_{n=2}^{M}(\|\nabla v_h^n\|^2 + \|\nabla w_h^n\|^2)$$
$$\le \|v_h^0\|^2 + \|w_h^0\|^2 + \|2v_h^1 - v_h^0\|^2 + \|2w_h^1 - w_h^0\|^2 + (\nu + \nu_m)\Delta t(\|\nabla v_h^1\|^2 + \|\nabla w_h^1\|^2 + 2\|\nabla v_h^0\|^2 + 2\|\nabla w_h^0\|^2) + \frac{4\Delta t}{\alpha}\sum_{n=1}^{M}(\|f_1(t^n)\|_{-1}^2 + \|f_2(t^n)\|_{-1}^2).$$

Remark 3.3. *Since Algorithm 3.1 is linear at each timestep and finite dimensional, the stability bound above is sufficient to provide well-posedness of the scheme. Uniqueness follows due to linearity, since the bounds on the difference between two solutions follow exactly as for the stability bound, but with a zero right hand side. Since the scheme is finite dimensional and linear at each time step, uniqueness implies existence, and thus solutions to Algorithm 3.1 must exist uniquely. That the unique solutions are bounded continuously by the data is given in the stability bound above.*

Proof. Choose $\chi_h = v_h^{n+1} \in V_h$ and $l_h = w_h^{n+1} \in V_h$ in (3.1)-(3.2). Then the trilinear terms vanish, leaving

$$\frac{1}{2\Delta t}(3v_h^{n+1} - 4v_h^n + v_h^{n-1}, v_h^{n+1}) + \frac{\nu+\nu_m}{2}\|\nabla v_h^{n+1}\|^2 + \frac{\nu-\nu_m}{2}((1+\theta)\nabla w_h^n - \theta\nabla w_h^{n-1}, \nabla v_h^{n+1}) = (f_1(t^{n+1}), v_h^{n+1})$$

and

$$\frac{1}{2\Delta t}(3w_h^{n+1} - 4w_h^n + w_h^{n-1}, w_h^{n+1}) + \frac{\nu+\nu_m}{2}\|\nabla w_h^{n+1}\|^2 + \frac{\nu-\nu_m}{2}((1+\theta)\nabla v_h^n - \theta\nabla v_h^{n-1}, \nabla w_h^{n+1}) = (f_2(t^{n+1}), w_h^{n+1}).$$

Adding these equations and using the identity

$$(3a - 4b + c, a) = \frac{a^2 + (2a-b)^2}{2} - \frac{b^2 + (2b-c)^2}{2} + \frac{(a-2b+c)^2}{2}, \tag{3.3}$$

we obtain

$$\begin{aligned}
&\frac{1}{4\Delta t}\big(\|v_h^{n+1}\|^2 - \|v_h^n\|^2 + \|2v_h^{n+1} - v_h^n\|^2 - \|2v_h^n - v_h^{n-1}\|^2 + \|w_h^{n+1}\|^2 - \|w_h^n\|^2 + \|2w_h^{n+1} - w_h^n\|^2 \\
&- \|2w_h^n - w_h^{n-1}\|^2 + \|v_h^{n+1} - 2v_h^n + v_h^{n-1}\|^2 + \|w_h^{n+1} - 2w_h^n + w_h^{n-1}\|^2\big) + \frac{\nu+\nu_m}{2}\big(\|\nabla v_h^{n+1}\|^2 + \|\nabla w_h^{n+1}\|^2\big) \\
&+ \frac{\nu-\nu_m}{2}((1+\theta)\nabla w_h^n - \theta\nabla w_h^{n-1}, \nabla v_h^{n+1}) + \frac{\nu-\nu_m}{2}((1+\theta)\nabla v_h^n - \theta\nabla v_h^{n-1}, \nabla w_h^{n+1}) \\
&= (f_1(t^{n+1}), v_h^{n+1}) + (f_2(t^{n+1}), w_h^{n+1}).
\end{aligned} \tag{3.4}$$

Applying Cauchy-Schwarz and Young's inequalities to the $(\nu-\nu_m)$ terms and dropping non-negative terms from the left hand side provides the bound

$$\begin{aligned}
&\frac{1}{4\Delta t}\big(\|v_h^{n+1}\|^2 - \|v_h^n\|^2 + \|2v_h^{n+1} - v_h^n\|^2 - \|2v_h^n - v_h^{n-1}\|^2 + \|w_h^{n+1}\|^2 - \|w_h^n\|^2 + \|2w_h^{n+1} - w_h^n\|^2 - \|2w_h^n - w_h^{n-1}\|^2\big) \\
&+ \frac{\nu+\nu_m}{2}\big(\|\nabla v_h^{n+1}\|^2 + \|\nabla w_h^{n+1}\|^2\big) \le \frac{|\nu-\nu_m|}{4}(1+\theta)\big(\|\nabla w_h^n\|^2 + \|\nabla v_h^{n+1}\|^2 + \|\nabla v_h^n\|^2 + \|\nabla w_h^{n+1}\|^2\big) \\
&+ \frac{|\nu-\nu_m|}{4}\theta\big(\|\nabla w_h^{n-1}\|^2 + \|\nabla v_h^{n+1}\|^2 + \|\nabla v_h^{n-1}\|^2 + \|\nabla w_h^{n+1}\|^2\big) \\
&+ \|f_1(t^{n+1})\|_{-1}\|\nabla v_h^{n+1}\| + \|f_2(t^{n+1})\|_{-1}\|\nabla w_h^{n+1}\|.
\end{aligned}$$

Next, we apply Young's inequality using α with the forcing terms, rearrange, and noting that $\alpha > 0$ by the assumed choice of θ,

$$\begin{aligned}
&\frac{1}{4\Delta t}\big(\|v_h^{n+1}\|^2 - \|v_h^n\|^2 + \|2v_h^{n+1} - v_h^n\|^2 - \|2v_h^n - v_h^{n-1}\|^2 + \|w_h^{n+1}\|^2 - \|w_h^n\|^2 + \|2w_h^{n+1} - w_h^n\|^2 - \|2w_h^n - w_h^{n-1}\|^2\big) \\
&+ \frac{\nu+\nu_m}{2}\big(\|\nabla v_h^{n+1}\|^2 + \|\nabla w_h^{n+1}\|^2\big) \le \frac{|\nu-\nu_m|}{4}(1+\theta)\big(\|\nabla w_h^n\|^2 + \|\nabla v_h^n\|^2\big) \\
&+ \frac{|\nu-\nu_m|}{4}\theta\big(\|\nabla w_h^{n-1}\|^2 + \|\nabla v_h^{n-1}\|^2\big) + \frac{\alpha + |\nu-\nu_m|(1+2\theta)}{4}\left(\|\nabla v_h^{n+1}\|^2 + \|\nabla w_h^{n+1}\|^2\right) \\
&+ \frac{1}{\alpha}\left(\|f_1(t^{n+1})\|_{-1} + \|f_2(t^{n+1})\|_{-1}\right).
\end{aligned}$$

Hiding terms on the left hand side, and adding and subtracting terms appropriately, we obtain

$$\frac{1}{4\Delta t}\big(\|v_h^{n+1}\|^2-\|v_h^n\|^2+\|2v_h^{n+1}-v_h^n\|^2-\|2v_h^n-v_h^{n-1}\|^2+\|w_h^{n+1}\|^2-\|w_h^n\|^2+\|2w_h^{n+1}-w_h^n\|^2 - \|2w_h^n-w_h^{n-1}\|^2\big)+\frac{\nu+\nu_m}{4}\big(\|\nabla v_h^{n+1}\|^2-\|\nabla v_h^n\|^2+\|\nabla w_h^{n+1}\|^2-\|\nabla w_h^n\|^2\big) + \frac{\nu+\nu_m-|\nu-\nu_m|(1+\theta)}{4}\big(\|\nabla v_h^n\|^2-\|\nabla v_h^{n-1}\|^2+\|\nabla w_h^n\|^2-\|\nabla w_h^{n-1}\|^2\big) + \frac{\nu+\nu_m-|\nu-\nu_m|(1+2\theta)}{4}\big(\|\nabla v_h^{n-1}\|^2+\|\nabla w_h^{n-1}\|^2\big)\le\frac{1}{\alpha}\left(\|f_1(t^{n+1})\|_{-1}^2+\|f_2(t^{n+1})\|_{-1}^2\right). \quad (3.5)$$

Now multiplying both sides by $4\Delta t$ and summing over time steps $n=1,\cdots,M-1$, we get

$$\|v_h^M\|^2+\|2v_h^M-v_h^{M-1}\|^2+\|w_h^M\|^2+\|2w_h^M-w_h^{M-1}\|^2+\alpha\Delta t\sum_{n=2}^{M}(\|\nabla v_h^n\|^2+\|\nabla w_h^n\|^2)\le\|v_h^0\|^2+\|w_h^0\|^2 + \|2v_h^1-v_h^0\|^2+\|2w_h^1-w_h^0\|^2+(\nu+\nu_m)\Delta t(\|\nabla v_h^1\|^2+\|\nabla w_h^1\|^2+2\|\nabla v_h^0\|^2+2\|\nabla w_h^0\|^2) + \frac{4\Delta t}{\alpha}\sum_{n=1}^{M}(\|f_1(t^n)\|_{-1}^2+\|f_2(t^n)\|_{-1}^2), \quad (3.6)$$

which finishes the proof. □

3.2 Convergence

We now consider convergence of the proposed decoupled, unconditionally stable scheme. Since the method departs from a second order framework when $\theta>1$, we do not expect a second order in time result. However, we are able to prove the method is nearly second order in practice; that is, in the typical case that ν and ν_m are small, the second order temporal error will be the dominant source of temporal error. Spatial convergence is found to be optimal.

Theorem 3.1. *For (v,w,p) satisfying (1.5)-(1.8) with regularity assumptions $v,w\in L^\infty(0,T;H^{k+1}(\Omega))$, $v_t,w_t,v_{tt},w_{tt}\in L^\infty(0,T;H^1(\Omega))$, and $v_{ttt},w_{ttt}\in L^\infty(0,T;L^2(\Omega))$, then the solution (v_h,w_h) to the Algorithm (3.1) converges unconditionally to the true solution: for any $\Delta t>0$,*

$$\|v(T)-v_h^M\|+\|w(T)-w_h^M\|+2\alpha\Delta t\sum_{n=2}^{M}\left(\|\nabla(v(t^n)-v_h^n)\|^2+\|\nabla(w(t^n)-w_h^n)\|^2\right)^{\frac12} \le C\left(h^k+(\Delta t)^2+(1-\theta)|\nu-\nu_m|\Delta t\right)$$

Proof. We start our proof by obtaining the error equations. At time level t^{n+1}, the continuous variational formulations of (1.5) and (1.8) can be written as

$$\left(\frac{3v(t^{n+1})-4v(t^n)+v(t^{n-1})}{2\Delta t},\chi_h\right)+\frac{\nu+\nu_m}{2}\left(\nabla v(t^{n+1}),\nabla\chi_h\right) + b^*\left(w(t^{n+1})-2w(t^n)+w(t^{n-1}),v(t^{n+1}),\chi_h\right)+b^*\left(2w(t^n)-w(t^{n-1}),v(t^{n+1}),\chi_h\right) +\frac{\nu-\nu_m}{2}\left(\nabla(w(t^{n+1})-(1+\theta)w(t^n)+\theta w(t^{n-1})),\nabla\chi_h\right)+\frac{\nu-\nu_m}{2}\left((1+\theta)\nabla w(t^n)-\theta\nabla w(t^{n-1}),\nabla\chi_h\right) =(f_1(t^{n+1}),\chi_h)-\left(v_t(t^{n+1})-\frac{3v(t^{n+1})-4v(t^n)+v(t^{n-1})}{2\Delta t},\chi_h\right), \quad (3.7)$$

and

$$\left(\frac{3w(t^{n+1})-4w(t^n)+w(t^{n-1})}{2\Delta t},l_h\right)+\frac{\nu+\nu_m}{2}\left(\nabla w(t^{n+1}),\nabla l_h\right)$$
$$+b*\left(v(t^{n+1})-2v(t^n)+v(t^{n-1}),w(t^{n+1}),l_h\right)+b^*\left(2v(t^n)-v(t^{n-1}),w(t^{n+1}),l_h\right)$$
$$+\frac{\nu-\nu_m}{2}\left(\nabla(v(t^{n+1})-(1+\theta)v(t^n)+\theta v(t^{n-1})),\nabla l_h\right)+\frac{\nu-\nu_m}{2}\left((1+\theta)\nabla v(t^n)-\theta\nabla v(t^{n-1}),\nabla l_h\right)$$
$$=(f_2(t^{n+1}),l_h)-\left(w_t(t^{n+1})-\frac{3w(t^{n+1})-4w(t^n)+w(t^{n-1})}{2\Delta t},l_h\right), \quad (3.8)$$

for all $\chi_h, l_h \in V_h$. Denote the errors by $e_v^n := v(t^n)-v_h^n$ and $e_w^n := w(t^n)-w_h^n$. Subtracting (3.1) and (3.2) from (3.7) and (3.8) respectively, provides

$$\left(\frac{3e_v^{n+1}-4e_v^n+e_v^{n-1}}{2\Delta t},\chi_h\right)+\frac{\nu+\nu_m}{2}\left(\nabla e_v^{n+1},\nabla\chi_h\right)+\frac{\nu-\nu_m}{2}\left((1+\theta)\nabla e_w^n-\theta\nabla e_w^{n-1},\nabla\chi_h\right)$$
$$+b^*(2e_w^n-e_w^{n-1},v(t^{n+1}),\chi_h)+b^*(2w_h^n-w_h^{n-1},e_v^{n+1},\chi_h)=-G_1(t,v,w,\chi_h), \quad (3.9)$$

and

$$\left(\frac{3e_w^{n+1}-4e_w^n+e_w^{n-1}}{2\Delta t},l_h\right)+\frac{\nu+\nu_m}{2}\left(\nabla e_w^{n+1},\nabla l_h\right)+\frac{\nu-\nu_m}{2}\left((1+\theta)\nabla e_v^n-\theta\nabla e_v^{n-1},\nabla l_h\right)$$
$$+b^*(2e_v^n-e_v^{n-1}\cdot\nabla w(t^{n+1}),l_h)+b^*(2v_h^n-v_h^{n-1},e_w^{n+1},l_h)=-G_2(t,v,w,l_h), \quad (3.10)$$

where

$$G_1(t,v,w,\chi_h):=\frac{\nu-\nu_m}{2}\left(\nabla(w(t^{n+1})-(1+\theta)w(t^n)+\theta w(t^{n-1})),\nabla\chi_h\right)$$
$$+b^*\left(w(t^{n+1})-2w(t^n)+w(t^{n-1}),v(t^{n+1}),\chi_h\right)+\left(v_t(t^{n+1})-\frac{3v(t^{n+1})-4v(t^n)+v(t^{n-1})}{2\Delta t},\chi_h\right)$$

and

$$G_2(t,v,w,l_h):=\frac{\nu-\nu_m}{2}\left(\nabla(v(t^{n+1})-(1+\theta)v(t^n)+\theta v(t^{n-1})),\nabla l_h\right)$$
$$+b^*\left(v(t^{n+1})-2v(t^n)+v(t^{n-1}),w(t^{n+1}),l_h\right)+\left(w_t(t^{n+1})-\frac{3w(t^{n+1})-4w(t^n)+w(t^{n-1})}{2\Delta t},l_h\right).$$

Now we decompose the errors as

$$e_v^n := v(t^n)-v_h^n=(v(t^n)-\tilde{v}^n)-(v_h^n-\tilde{v}^n):=\eta_v^n-\phi_h^n,$$
$$e_w^n := w(t^n)-w_h^n=(w(t^n)-\tilde{w}^n)-(w_h^n-\tilde{w}^n):=\eta_w^n-\psi_h^n,$$

where $\tilde{v}^n=P_{V_h}^{L^2}(v(t^n))\in V_h$ and $\tilde{w}^n=P_{V_h}^{L^2}(w(t^n))\in V_h$ are the L^2 projections of $v(t^n)$ and $w(t^n)$ into V_h respectively. Note that $(\eta_v^n,v_h)=(\eta_w^n,v_h)=0\ \forall v_h\in V_h$. Rewriting, we have for χ_h,

$l_h \in V_h$

$$\left(\frac{3\phi_h^{n+1}-4\phi_h^n+\phi_h^{n-1}}{2\Delta t},\chi_h\right)+\frac{\nu+\nu_m}{2}\left(\nabla\phi_h^{n+1},\nabla\chi_h\right)+\frac{\nu-\nu_m}{2}\left((1+\theta)\nabla\psi_h^n-\theta\nabla\psi_h^{n-1},\nabla\chi_h\right)$$
$$+b^*\left(2\psi_h^n-\psi_h^{n-1},v(t^{n+1}),\chi_h\right)+b^*(2w_h^n-w_h^{n-1},\phi_h^{n+1},\chi_h)=\frac{\nu+\nu_m}{2}\left(\nabla\eta_v^{n+1},\nabla\chi_h\right)+G_1(t,v,w,\chi_h)$$
$$+\frac{\nu-\nu_m}{2}\left((1+\theta)\nabla\eta_w^n-\theta\nabla\eta_w^{n-1},\nabla\chi_h\right)+b^*\left(2\eta_w^n-\eta_w^{n-1},v(t^{n+1}),\chi_h\right)+b^*\left(2w_h^n-w_h^{n-1},\eta_v^{n+1},\chi_h\right),\tag{3.11}$$

and

$$\left(\frac{3\psi_h^{n+1}-4\psi_h^n+\psi_h^{n-1}}{2\Delta t},l_h\right)+\frac{\nu+\nu_m}{2}\left(\nabla\psi_h^{n+1},\nabla l_h\right)+\frac{\nu-\nu_m}{2}\left((1+\theta)\nabla\phi_h^n-\theta\nabla\phi_h^{n-1},\nabla l_h\right)$$
$$+b^*(2\phi_h^n-\phi_h^{n-1},w(t^{n+1}),l_h)+b^*(2v_h^n-v_h^{n-1},\psi_h^{n+1},l_h)=\frac{\nu+\nu_m}{2}\left(\nabla\eta_w^{n+1},\nabla l_h\right)+G_2(t,v,w,l_h)$$
$$+\frac{\nu-\nu_m}{2}\left((1+\theta)\nabla\eta_v^n-\theta\nabla\eta_v^{n-1},\nabla l_h\right)+b^*(2\eta_v^n-\eta_v^{n-1},w(t^{n+1}),l_h)+b^*(2v_h^n-v_h^{n-1},\eta_w^{n+1},l_h).\tag{3.12}$$

Choose $\chi_h=\phi_h^{n+1}$, $l_h=\psi_h^{n+1}$ and use the identity (3.3) in (3.11) and (3.12), to obtain

$$\frac{1}{4\Delta t}\left(\|\phi_h^{n+1}\|^2-\|\phi_h^n\|^2+\|2\phi_h^{n+1}-\phi_h^n\|^2-\|2\phi_h^n-\phi_h^{n-1}\|^2+\|\phi_h^{n+1}-2\phi_h^n+\phi_h^{n-1}\|^2\right)+\frac{\nu+\nu_m}{2}\|\nabla\phi_h^{n+1}\|^2$$
$$\leq(1+\theta)\frac{|\nu-\nu_m|}{2}\Big\{|\left(\nabla\eta_w^n,\nabla\phi_h^{n+1}\right)|+|(\nabla\psi_h^n,\nabla\phi_h^{n+1})|\Big\}+\theta\frac{|\nu-\nu_m|}{2}\Big\{|\left(\nabla\eta_w^{n-1},\nabla\phi_h^{n+1}\right)|$$
$$+|(\nabla\psi_h^{n-1},\nabla\phi_h^{n+1})|\Big\}+\frac{\nu+\nu_m}{2}|\left(\nabla\eta_v^{n+1},\nabla\phi_h^{n+1}\right)|+|b^*\left(2\eta_w^n-\eta_w^{n-1},v(t^{n+1}),\phi_h^{n+1}\right)|$$
$$+|b^*\left(2w_h^n-w_h^{n-1},\eta_v^{n+1},\phi_h^{n+1}\right)|+|b^*\left(2\psi_h^n-\psi_h^{n-1},v(t^{n+1}),\phi_h^{n+1}\right)|+|G_1(t,v,w,\phi_h^{n+1})|,\tag{3.13}$$

and

$$\frac{1}{4\Delta t}\left(\|\psi_h^{n+1}\|^2-\|\psi_h^n\|^2+\|2\psi_h^{n+1}-\psi_h^n\|^2-\|2\psi_h^n-\psi_h^{n-1}\|^2+\|\psi_h^{n+1}-2\psi_h^n+\psi_h^{n-1}\|^2\right)+\frac{\nu+\nu_m}{2}\|\nabla\psi_h^{n+1}\|^2$$
$$\leq(1+\theta)\frac{|\nu-\nu_m|}{2}\Big\{|\left(\nabla\eta_v^n,\nabla\psi_h^{n+1}\right)|+|(\nabla\phi_h^n,\nabla\psi_h^{n+1})|\Big\}+\theta\frac{|\nu-\nu_m|}{2}\Big\{|\left(\nabla\eta_v^{n-1},\nabla\psi_h^{n+1}\right)|$$
$$+|(\nabla\phi_h^{n-1},\nabla\psi_h^{n+1})|\Big\}+\frac{\nu+\nu_m}{2}|\left(\nabla\eta_w^{n+1},\nabla\psi_h^{n+1}\right)|+|b^*\left(2\eta_v^n-\eta_v^{n-1},w(t^{n+1}),\psi_h^{n+1}\right)|$$
$$+|b^*\left(2v_h^n-v_h^{n-1},\eta_w^{n+1},\psi_h^{n+1}\right)|+|b^*\left(2\phi_h^n-\phi_h^{n-1},w(t^{n+1}),\psi_h^{n+1}\right)|+|G_2(t,v,w,\psi_h^{n+1})|\tag{3.14}$$

We now turn our attention to finding bounds on the right side terms of(3.13) (the estimates for (3.14) are similar). Applying Cauchy-Schwarz and Young's inequalities on the first five terms results in

$$
\begin{aligned}
(1+\theta)\frac{|\nu-\nu_m|}{2}|(\nabla\psi_h^n,\nabla\phi_h^{n+1})| &\le (1+\theta)\frac{|\nu-\nu_m|}{4}\left(\|\nabla\psi_h^n\|^2+\|\nabla\phi_h^{n+1}\|^2\right),\\
\theta\frac{|\nu-\nu_m|}{2}|(\nabla\psi_h^{n-1},\nabla\phi_h^{n+1})| &\le \theta\frac{|\nu-\nu_m|}{4}\left(\|\nabla\phi_h^{n+1}\|^2+\|\nabla\psi_h^{n-1}\|^2\right),\\
(1+\theta)\frac{|\nu-\nu_m|}{2}|\left(\nabla\eta_w^n,\nabla\phi_h^{n+1}\right)| &\le \frac{\alpha}{28}\|\nabla\phi_h^{n+1}\|^2+\frac{7(1+\theta)^2(\nu-\nu_m)^2}{4\alpha}\|\nabla\eta_w^n\|^2,\\
\theta\frac{|\nu-\nu_m|}{2}|\left(\nabla\eta_w^{n-1},\nabla\phi_h^{n+1}\right)| &\le \frac{\alpha}{28}\|\nabla\phi_h^{n+1}\|^2+\frac{7\theta^2(\nu-\nu_m)^2}{4\alpha}\|\nabla\eta_w^{n-1}\|^2,\\
\frac{\nu+\nu_m}{2}|\left(\nabla\eta_v^{n+1},\nabla\phi_h^{n+1}\right)| &\le \frac{\alpha}{28}\|\nabla\phi_h^{n+1}\|^2+\frac{7(\nu+\nu_m)^2}{4\alpha}\|\nabla\eta_v^{n+1}\|^2
\end{aligned}
$$

Applying Hölder and Young's inequalities with (2.1) on the first two nonlinear terms yields

$$
\begin{aligned}
|b^*\left(2\eta_w^n-\eta_w^{n-1},v(t^{n+1}),\phi_h^{n+1}\right)| &\le C\|\nabla(2\eta_w^n-\eta_w^{n-1})\|\|\nabla v(t^{n+1})\|\|\nabla\phi_h^{n+1}\|\\
&\le \frac{\alpha}{28}\|\nabla\phi_h^{n+1}\|^2+\frac{7C}{\alpha}\|\nabla v(t^{n+1})\|^2\|\nabla(2\eta_w^n-\eta_w^{n-1})\|^2,\\
|b^*\left(2w_h^n-w_h^{n-1},\eta_v^{n+1},\phi_h^{n+1}\right)| &\le C\|\nabla(2w_h^n-w_h^{n-1})\|\|\nabla\eta_v^{n+1}\|\|\nabla\phi_h^{n+1}\|\\
&\le \frac{\alpha}{28}\|\nabla\phi_h^{n+1}\|^2+\frac{7C}{\alpha}\|\nabla(2w_h^n-w_h^{n-1})\|^2\|\nabla\eta_v^{n+1}\|^2.
\end{aligned}
$$

For the third nonlinear term, we use Hölder's inequality, Sobolev embedding theorems, Poincare's and Young's inequalities to reveal

$$
\begin{aligned}
|\left((2\psi_h^n-\psi_h^{n-1})\cdot\nabla v(t^{n+1}),\phi_h^{n+1}\right)| &\le C\|2\psi_h^n-\psi_h^{n-1}\|\|\nabla v(t^{n+1})\|_{L^6}\|\phi_h^{n+1}\|_{L^3}\\
&\le C\|2\psi_h^n-\psi_h^{n-1}\|\|v(t^{n+1})\|_{H^2}\|\phi_h^n\|^{1/2}\|\nabla\phi_h^{n+1}\|^{1/2}\\
&\le C\|2\psi_h^n-\psi_h^{n-1}\|\|v(t^{n+1})\|_{H^2}\|\nabla\phi_h^{n+1}\|\\
&\le \frac{\alpha}{28}\|\nabla\phi_h^{n+1}\|^2+\frac{7C}{\alpha}\|v(t^{n+1})\|_{H^2}^2\|2\psi_h^n-\psi_h^{n-1}\|^2.
\end{aligned}
$$

Using Taylor's series, Cauchy-Schwarz and Young's inequalities the last term is evaluated as

$$
\begin{aligned}
|G_1(t,v,w,\chi_h)| \le{}& C(\Delta t)^4\left(\|v_{ttt}(t^*)\|^2+\|\nabla w_{tt}(t^{**})\|^2\|\nabla v(t^{n+1})\|^2\right)\\
&+\frac{7(\nu-\nu_m)^2(1-\theta)^2(\Delta t)^2}{4\alpha}\|\nabla w_t(t^{***})\|^2+\frac{\alpha}{28}\|\nabla\phi_h^{n+1}\|^2,
\end{aligned}
$$

with $t^*,t^{**},t^{***}\in[t^{n-1},t^{n+1}]$. Using these estimates in (3.13) and reducing produces

$$
\begin{aligned}
&\frac{1}{4\Delta t}\left(\|\phi_h^{n+1}\|^2-\|\phi_h^n\|^2+\|2\phi_h^{n+1}-\phi_h^n\|^2-\|2\phi_h^n-\phi_h^{n-1}\|^2\right)+\frac{\alpha+\nu+\nu_m}{4}\|\nabla\phi_h^{n+1}\|^2 \le \theta\frac{|\nu-\nu_m|}{4}\|\nabla\psi_h^{n-1}\|^2\\
&+(1+\theta)\frac{|\nu-\nu_m|}{4}\|\nabla\psi_h^n\|^2+\frac{7(1+\theta)^2(\nu-\nu_m)^2}{4\alpha}\|\nabla\eta_w^n\|^2+\frac{7\theta^2(\nu-\nu_m)^2}{4\alpha}\|\nabla\eta_w^{n-1}\|^2+\frac{7(\nu+\nu_m)^2}{4\alpha}\|\nabla\eta_v^{n+1}\|^2\\
&+\frac{7C}{\alpha}\|\nabla v(t^{n+1})\|^2\|\nabla(2\eta_w^n-\eta_w^{n-1})\|^2+\frac{7C}{\alpha}\|\nabla(2w_h^n-w_h^{n-1})\|^2\|\nabla\eta_v^{n+1}\|^2+\frac{7C}{\alpha}\|v(t^{n+1})\|_{H^2}^2\|2\psi_h^n-\psi_h^{n-1}\|^2\\
&\quad+C(\Delta t)^4\left(\|v_{ttt}(t^*)\|^2+\|\nabla w_{tt}(t^{**})\|^2\|\nabla v(t^{n+1})\|^2\right)+\frac{7(\nu-\nu_m)^2(1-\theta)^2(\Delta t)^2}{4\alpha}\|\nabla w_t(t^{***})\|^2.
\end{aligned}
\tag{3.15}
$$

Apply similar techniques to (3.14), we get

$$\frac{1}{4\Delta t}\left(\|\psi_h^{n+1}\|^2-\|\psi_h^n\|^2+\|2\psi_h^{n+1}-\psi_h^n\|^2-\|2\psi_h^n-\psi_h^{n-1}\|^2\right)+\frac{\alpha+\nu+\nu_m}{4}\|\nabla\psi_h^{n+1}\|^2 \le \theta\frac{|\nu-\nu_m|}{4}\|\nabla\phi_h^{n-1}\|^2$$
$$+(1+\theta)\frac{|\nu-\nu_m|}{4}\|\nabla\phi_h^n\|^2+\frac{7(1+\theta)^2(\nu-\nu_m)^2}{4\alpha}\|\nabla\eta_v^n\|^2+\frac{7\theta^2(\nu-\nu_m)^2}{4\alpha}\|\nabla\eta_v^{n-1}\|^2+\frac{7(\nu+\nu_m)^2}{4\alpha}\|\nabla\eta_w^{n+1}\|^2$$
$$+\frac{7C}{\alpha}\|\nabla w(t^{n+1})\|^2\|\nabla(2\eta_v^n-\eta_v^{n-1})\|^2+\frac{7C}{\alpha}\|\nabla(2v_h^n-v_h^{n-1})\|^2\|\nabla\eta_w^{n+1}\|^2+\frac{7C}{\alpha}\|w(t^{n+1})\|_{H^2}^2\|2\phi_h^n-\phi_h^{n-1}\|^2$$
$$+C(\Delta t)^4\left(\|w_{ttt}(s^*)\|^2+\|\nabla v_{tt}(s^{**})\|^2\|\nabla w(t^{n+1})\|^2\right)+\frac{7(\nu-\nu_m)^2(1-\theta)^2(\Delta t)^2}{4\alpha}\|\nabla v_t(s^{***})\|^2, \tag{3.16}$$

with $s^*, s^{**}, s^{***} \in [t^{n-1}, t^{n+1}]$. Now add equations (3.15) and (3.16), multiply by $4\Delta t$, use regularity assumptions, $\|\phi_h^0\| = \|\psi_h^0\| = \|\phi_h^1\| = \|\psi_h^1\| = 0$, $\Delta t M = T$, and sum over the time steps to find

$$\|\phi_h^M\|^2+\|2\phi_h^M-\phi_h^{M-1}\|^2+\|\psi_h^M\|^2+\|2\psi_h^M-\psi_h^{M-1}\|^2+2\alpha\Delta t\sum_{n=2}^{M}\left(\|\nabla\phi_h^n\|^2+\|\nabla\psi_h^n\|^2\right)$$
$$\le C\Delta t\sum_{n=0}^{M}\left(\|\nabla\eta_v^n\|^2+\|\nabla\eta_w^n\|^2\right)+C\left((\Delta t)^4+(\nu-\nu_m)^2(1-\theta)^2(\Delta t)^2\right)$$
$$+C\Delta t\sum_{n=1}^{M-1}\left(\|\nabla\left(2w_h^n-w_h^{n-1}\right)\|^2\|\nabla\eta_v^{n+1}\|^2+\|\nabla\left(2v_h^n-v_h^{n-1}\right)\|^2\|\nabla\eta_w^{n+1}\|^2\right)$$
$$+C\Delta t\sum_{n=1}^{M-1}\left(\|w(t^{n+1})\|_{L^\infty(0,T;H^2(\Omega))}^2\|2\phi_h^n-\phi_h^{n-1}\|^2+\|v(t^{n+1})\|_{L^\infty(0,T;H^2(\Omega))}^2\|2\psi_h^n-\psi_h^{n-1}\|^2\right). \tag{3.17}$$

Applying the discrete Gronwall lemma and bounds for $\|\nabla\eta_v\|$ and $\|\nabla\eta_w\|$, we have for any $\Delta t > 0$ that

$$\|\phi_h^M\|^2+\|\psi_h^M\|^2+2\alpha\Delta t\sum_{n=2}^{M}\left(\|\nabla\phi_h^n\|^2+\|\nabla\psi_h^n\|^2\right)\le C\left(h^{2k}+(\Delta t)^4+(\nu-\nu_m)^2(1-\theta)^2(\Delta t)^2\right).$$

Now using the triangle inequality completes the proof. □

4 Numerical Experiments

In this section we perform three numerical experiments: a test of stability with varying θ, a verification of convergence rates, and simulation of MHD channel flow past a step. For the first two tests, we use the test problem with analytical solution

$$v=\begin{pmatrix}\cos y+(1+e^t)\sin y\\ \sin x+(1+e^t)\cos x\end{pmatrix}, w=\begin{pmatrix}\cos y-(1+e^t)\sin y\\ \sin x-(1+e^t)\cos x\end{pmatrix} \quad p=-\lambda=\sin(x+y)(1+e^t),$$

on the domain $\Omega = (0,1)^2$. The forcings f_1 and f_2 are calculated from the true solution, the values of ν and ν_m, and the initial conditions and boundary conditions use the analytical solution. All simulations were run using the software Freefem++ [19], and (P_2, P_1^{disc}) Scott-Vogelius elements on barycenter refined triangular meshes.

4.1 Numerical experiment 1: Testing stability versus θ

For our first numerical test, we consider stability of the proposed algorithm for varying θ, using the test problem described above with $\nu = 1$ and $\nu_m = 0.1$. We simulate until T=1 using Algorithm 3.1 with $h = 1/64$, $\Delta t = 1/256$, and three choices of θ: $\theta = 1$ (the BDF2 case), $\theta = 0.167$ and $\theta = \theta_{critical} = 0.111$. Our theoretical results prove that the scheme is stable for $\theta < \theta_{critical}$, and suggest the scheme is unstable for larger θ.

Figure 1 shows plots of $\frac{1}{2}\|\nabla v_h^n\|^2$ and $\frac{1}{2}\|\nabla w_h^n\|^2$ with time, for each of the θ values. The solution norms remain stable for $\theta = \theta_{critical} = 0.111$. However, for both cases of $\theta > \theta_{critical}$, we observe solution blowup / instability. In particular, for the BDF2 case ($\theta = 1$), the blowup occurs very quickly.

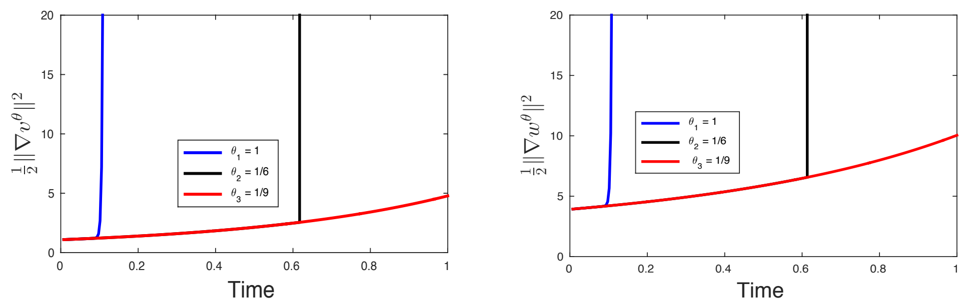

Figure 1: Plots of $\frac{1}{2}\|\nabla v_h\|^2$ and $\frac{1}{2}\|\nabla w_h\|^2$ versus time, for numerical experiment 1. Only the case of $\theta = \theta_{critical}$ remains stable.

4.2 Numerical experiment 2: Convergence rate verification

Next, we test the theoretical convergence rates predicted by the theory. Here, we use the same analytical test problem as the first numerical example, but now with $\nu = 0.001$ and $\nu_m = 0.01$, $\theta = \theta_{critical} = \frac{1}{9}$, and (P_2, P_1^{disc}) Scott-Vogelius elements on barycenter refined triangular meshes. Spatial and temporal convergence rates are calculated, and from the theory we expect $O(h^2 + \Delta t^2 + (1-\theta)|\nu - \nu_m|\Delta t)$ convergence. For spatial convergence testing, we select a very small endtime $T = 0.001$, fix $\Delta t = \frac{T}{8}$, and then compute on successively refined uniform meshes. For temporal convergence, we fix $h = 1/64$, $T = 1$, and compute with successively refined time step sizes.

Errors and rates are shown in table 1 for v, and we omit the w results since they are very similar. From the tables, we observe second order spatial convergence as expected. For temporal convergence, we also observe a rate near 2. We also compute errors and rates for usual BDF2 ($\theta = 1$) as Δt is refined, and we observe from the tables that BDF2 error blows up as $\Delta t \to 0$; these terrible BDF2 results are expected since $1 \gg \theta_{critical}$.

4.3 Numerical experiment 3: MHD Channel Flow over a step

Our final experiment is to test the proposed method for MHD channel flow past a step. The problem setup follows the classical NSE benchmark [17], using $\Omega = (0, 30) \times (0, 10)$ with a 1×1 step placed five units into the channel on the bottom. We take T=40, $\Delta t = 0.025$, and full Dirichlet boundary conditions corresponding to no slip velocity on the walls, $u = < y(10-y)/25, 0 >^T$ on

Temporal convergence (fixed h=1/64)				
	$\theta = 1$(BDF2)		$\theta = 1/9$	
Δt	$\|v - v_h\|_{2,1}$	rate	$\|v - v_h\|_{2,1}$	rate
$\frac{T}{4}$	9.006e-2		7.410e-2	
$\frac{T}{8}$	3.625e-2	1.31	2.574e-2	1.53
$\frac{T}{16}$	9.298e-2	–	7.668e-3	1.75
$\frac{T}{32}$	4.995e+2	–	1.962e-3	1.97
$\frac{T}{64}$	5.217e+4	–	4.178e-4	2.23

Spatial convergence (fixed T=0.001, $\Delta t = \frac{T}{8}$)		
	$\theta = 1/9$	
h	$\|v - v_h\|_{2,1}$	rate
$\frac{1}{4}$	1.009e-4	
$\frac{1}{8}$	2.538e-5	1.99
$\frac{1}{16}$	6.363e-6	2.00
$\frac{1}{32}$	1.598e-6	1.99
$\frac{1}{64}$	4.014e-7	1.99

Table 1: Spatial and temporal convergence rates for $\nu = 0.01$, $\nu_m = 0.001$, using the critical $\theta = \frac{1}{9}$. Also shown is the blowup of error as $\Delta t \to 0$ when $\theta = 1$ (the usual BDF2 case).

the inlet and outlet, and $B =< 0, 1 >^T$ on all boundaries. The initial conditions corresponds to no magnetic field and a parabolic velocity profile $u_0 =< y(10 - y)/25, 0 >^T$. A coupling number of $s = 0.01$ is used in all the simulations, as is a Delaunay generated triangulation which provides 1,778,630 total degrees of freedom when used with (P_2, P_1^{disc}) Scott-Vogelius elements.

We show results for two cases below, the case $\nu = 0.001$ and $\nu_m = 1$ in figure 2, and $\nu = 0.001$ and $\nu_m = 0.1$ in figure 3. For each case, we ran simulations with $\theta = \theta_{critical}$, a somewhat larger θ, and also $\theta = 1$ (BDF2). The figures show plots of streamlines over speed contours, and magnetic field contours at T=40. Only the simulations with $\theta = \theta_{critical}$ remained stable and accurate to T=40. The simulations with larger θ are clearly very inaccurate, and exhibit spurious oscillations and instability.

5 Conclusion and future works

We proposed, analyzed, and tested a new, efficient scheme for MHD, and rigorously proved its unconditional stability, well-posedness, and convergence, under an appropriate choice of θ (which is made a priori, based on ν and ν_m). The proposed method may be an enabling tool for MHD simulations, since it stably decouples the MHD system into two Oseen problems at each timestep that can be solved simultaneously, converges optimally in space, and behaves like second order in time when ν and ν_m are small, all without any restriction on the time step size or on data ν and ν_m (which the full BDF2 method does require). The decoupling allows for the solving of potentially much bigger problems than primitive variable MHD algorithms can solve, since schemes in primitive variables require solving very large coupled linear systems (or excessively small time step sizes) for stable computations.

In addition to the possibility of more easily solving bigger MHD problems with the proposed method compared to fully coupled methods based on primitive variable formulations, it is worth exploring if the proposed scheme can likely be combined with recent stabilization ideas such as that in [32], for more accurate large scale simulations that don't have sufficient resolution to fully resolve all active scales.

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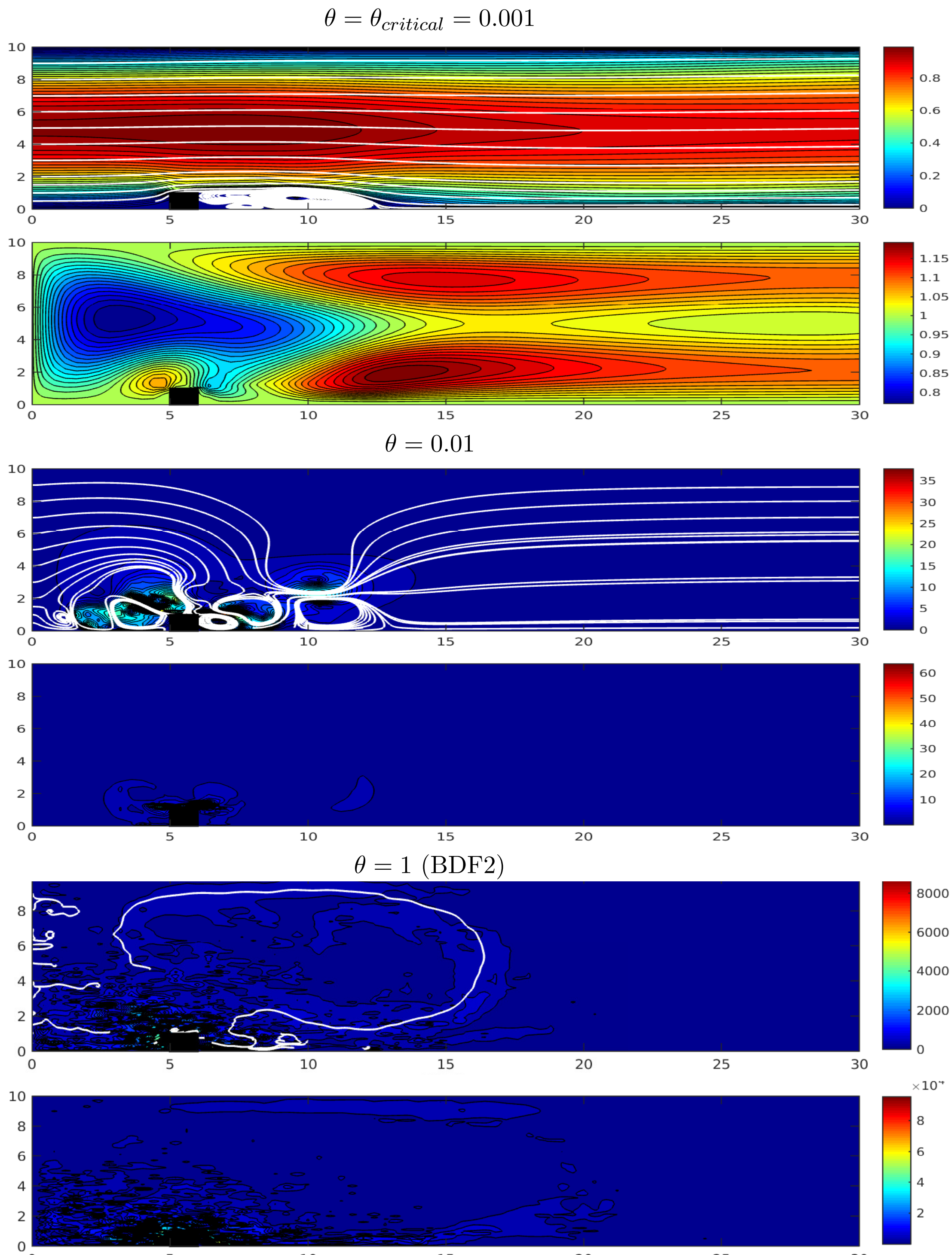

Figure 2: Velocity and magnetic field solutions at $T = 40$, for $s = 0.01$, $\nu = 0.001$ and $\nu_m = 1.0$, for varying θ. For $\theta = \theta_{critical}$, a stable and accurate solution is found, and unstable solutions are found for larger θ.

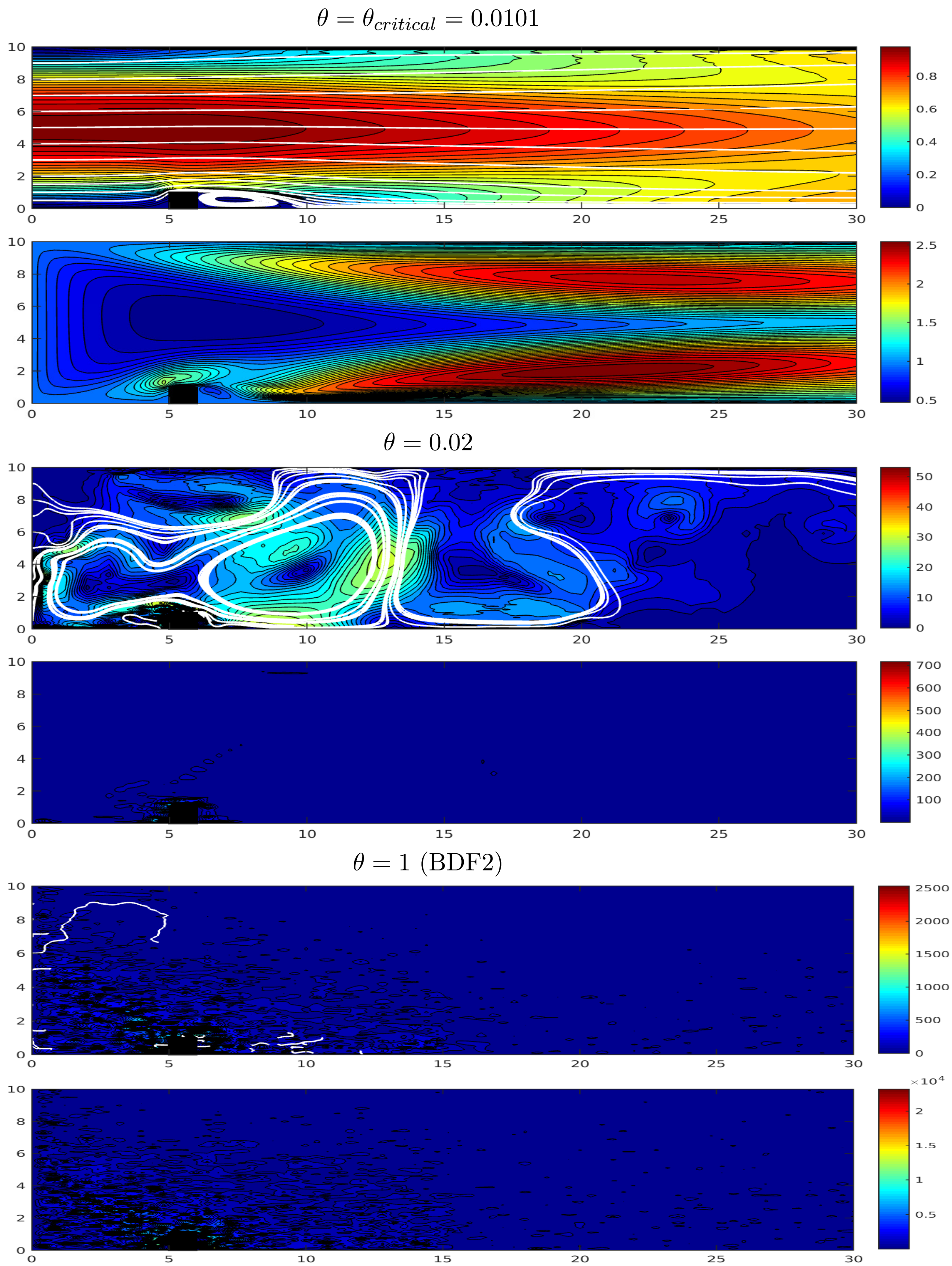

Figure 3: Velocity and magnetic field solutions at $T = 40$, for $s = 0.01$, $\nu = 0.001$ and $\nu_m = 0.1$, for varying θ. For $\theta = \theta_{critical}$, a stable and accurate solution is found, and unstable solutions are found for larger θ.

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7 Appendix

We prove here a conditional stability result for the full second order method, i.e. when $\theta = 1$, which does not assume $\frac{1}{2} < Pr_m < 2$. The condition is that $\Delta t \leq \frac{h^2(\nu+\nu_m-|\nu-\nu_m|)}{C_i(\nu-\nu_m)^2}$, where C_i an the inverse inequality constant, and thus if $\nu - \nu_m$ is not small (which is equivalent to Pr_m near 1), this can be a severe timestep restriction when fine meshes are used.

Lemma 7.1 (BDF2Stability)**.** *Consider Algorithm 3.1 with $\theta = 1$ (the full second order method). If the mesh is sufficiently regular so that the inverse inequality holds (with constant C_i) and the time step is chosen to satisfy*

$$\Delta t \leq \frac{h^2(\nu + \nu_m - |\nu - \nu_m|)}{C_i(\nu - \nu_m)^2},$$

then the method is stable and solutions satisfy

$$\|v_h^M\|^2 + \|w_h^M\|^2 + \frac{(\nu + \nu_m - |\nu - \nu_m|)\Delta t}{2} \sum_{n=1}^{M-1} \left(\|\nabla v_h^{n+1}\|^2 + \|\nabla w_h^{n+1}\|^2\right) \leq C(\nu, \nu_m, v_h^0, v_h^1, w_h^0, w_h^1, f_1, f_2).$$

Proof. Choose $\theta = 1$, $\chi_h = v_h^{n+1} \in V_h$ and $l_h = w_h^{n+1} \in V_h$ in Algorithm 3.1, (3.1)-(3.2). This vanishes the nonlinear and pressure terms, and leaves

$$\frac{1}{2\Delta t}(3v_h^{n+1} - 4v_h^n + v_h^{n-1}, v_h^{n+1}) + \frac{\nu + \nu_m}{2}\|\nabla v_h^{n+1}\|^2 + \frac{\nu - \nu_m}{2}(\nabla(2w_h^n - w_h^{n-1}), \nabla_+ z_h^{n+1}) = (f_1^{n+1}, v_h^{n+1}), \quad (7.1)$$

$$\frac{1}{2\Delta t}(3w_h^{n+1} - 4w_h^n + w_h^{n-1}, w_h^{n+1}) + \frac{\nu + \nu_m}{2}\|\nabla w_h^{n+1}\|^2 + \frac{\nu - \nu_m}{2}(\nabla(2v_h^n - v_h^{n-1}), \nabla w_h^{n+1}) = (f_2^{n+1}, w_h^{n+1}). \quad (7.2)$$

Using the usual BDF2 identity on the time derivative terms and adding the equations yields

$$\frac{1}{4\Delta t}\big(\|v_h^{n+1}\|^2 - \|v_h^n\|^2 + \|2v_h^{n+1} - v_h^n\|^2 - \|2v_h^n - v_h^{n-1}\|^2 + \|w_h^{n+1}\|^2 - \|w_h^n\|^2 + \|2w_h^{n+1} - w_h^n\|^2 - \|2w_h^n - w_h^{n-1}\|^2 + \|v_h^{n+1} - 2v_h^n + v_h^{n-1}\|^2 + \|w_h^{n+1} - 2w_h^n + w_h^{n-1}\|^2\big) + \frac{\nu + \nu_m}{2}\big(\|\nabla v_h^{n+1}\|^2 + \|\nabla w_h^{n+1}\|^2\big) + \frac{\nu - \nu_m}{2}(\nabla(2w_h^n - w_h^{n-1}), \nabla v_h^{n+1}) + \frac{\nu - \nu_m}{2}(\nabla(2v_h^n - v_h^{n-1}), \nabla w_h^{n+1}) = (f_1^{n+1}, v_h^{n+1}) + (f_2^{n+1}, w_h^{n+1}), \quad (7.3)$$

and then adding and subtracting the term $\frac{\nu-\nu_m}{2}\left(\nabla v_h^{n+1}, \nabla w_h^{n+1}\right)$ provides

$$\frac{1}{4\Delta t}\big(\|v_h^{n+1}\|^2 - \|v_h^n\|^2 + \|2v_h^{n+1} - v_h^n\|^2 - \|2v_h^n - v_h^{n-1}\|^2 + \|w_h^{n+1}\|^2 - \|w_h^n\|^2 + \|2w_h^{n+1} - w_h^n\|^2$$
$$-\|2w_h^n - w_h^{n-1}\|^2 + \|v_h^{n+1} - 2v_h^n + v_h^{n-1}\|^2 + \|w_h^{n+1} - 2w_h^n + w_h^{n-1}\|^2\big) + \frac{\nu+\nu_m}{2}\big(\|\nabla v_h^{n+1}\|^2 + \|\nabla w_h^{n+1}\|^2\big)$$
$$-\frac{\nu-\nu_m}{2}(\nabla(v_h^{n+1} - 2v_h^n + v_h^{n-1}), \nabla w_h^{n+1}) - \frac{\nu-\nu_m}{2}(\nabla(w_h^{n+1} - 2w_h^n + w_h^{n-1}), \nabla v_h^{n+1})$$
$$+\frac{\nu-\nu_m}{2}(\nabla w_h^{n+1}, \nabla v_h^{n+1}) + \frac{\nu-\nu_m}{2}(\nabla v_h^{n+1}, \nabla w_h^{n+1}) = (f_1^{n+1}, v_h^{n+1}) + (f_2^{n+1}, w_h^{n+1}). \quad (7.4)$$

Using Cauchy-Schwarz and Young's inequalities we have that

$$\frac{1}{4\Delta t}\big(\|v_h^{n+1}\|^2 - \|v_h^n\|^2 + \|2v_h^{n+1} - v_h^n\|^2 - \|2v_h^n - v_h^{n-1}\|^2 + \|w_h^{n+1}\|^2 - \|w_h^n\|^2 + \|2w_h^{n+1} - w_h^n\|^2$$
$$-\|2w_h^n - w_h^{n-1}\|^2 + \|v_h^{n+1} - 2v_h^n + v_h^{n-1}\|^2 + \|w_h^{n+1} - 2w_h^n + w_h^{n-1}\|^2\big) + \frac{\nu+\nu_m}{2}\big(\|\nabla v_h^{n+1}\|^2 + \|\nabla w_h^{n+1}\|^2\big)$$
$$\leq \frac{|\nu-\nu_m|}{2}\|\nabla(w_h^{n+1} - 2w_h^n + w_h^{n-1})\|\|\nabla v_h^{n+1}\| + \frac{|\nu-\nu_m|}{2}|\|\nabla(v_h^{n+1} - 2v_h^n + v_h^{n-1})\|\|\nabla w_h^{n+1}\|$$
$$+|\nu-\nu_m|\|\nabla w_h^{n+1}\|\|\nabla v_h^{n+1}\| + \|f_1^{n+1}\|_{-1}\|\nabla v_h^{n+1}\| + \|f_2^{n+1}\|_{-1}\|\nabla w_h^{n+1}\|. \quad (7.5)$$

Young's inequality provides the following bounds on the last five terms in (7.5):

$$|\nu-\nu_m|\|\nabla v_h^{n+1}\|\|\nabla w_h^{n+1}\| \leq \frac{|\nu-\nu_m|}{2}\|\nabla v_h^{n+1}\|^2 + \frac{|\nu-\nu_m|}{2}\|\nabla w_h^{n+1}\|^2,$$
$$\|f_1^{n+1}\|_{-1}\|\nabla v_h^{n+1}\| \leq \frac{\nu+\nu_m-|\nu-\nu_m|}{8}\|\nabla v_h^{n+1}\|^2 + \frac{2}{\nu+\nu_m-|\nu-\nu_m|}\|f_1^{n+1}\|_{-1}^2,$$
$$\|f_2^{n+1}\|_{-1}\|\nabla w_h^{n+1}\| \leq \frac{\nu+\nu_m-|\nu-\nu_m|}{8}\|\nabla w_h^{n+1}\|^2 + \frac{2}{\nu+\nu_m-|\nu-\nu_m|}\|f_2^{n+1}\|_{-1}^2,$$

$$\frac{|\nu-\nu_m|}{2}|\|\nabla(w_h^{n+1} - 2w_h^n + w_h^{n-1})\|\|\nabla v_h^{n+1}\| \leq \frac{\nu+\nu_m-|\nu-\nu_m|}{4}\|\nabla v_h^{n+1}\|^2$$
$$+\frac{(\nu-\nu_m)^2}{4(\nu+\nu_m-|\nu-\nu_m|)}\|\nabla(w_h^{n+1} - 2w_h^n + w_h^{n-1})\|^2,$$

$$\frac{|\nu-\nu_m|}{2}|\|\nabla(v_h^{n+1} - 2v_h^n + v_h^{n-1})\|\|\nabla w_h^{n+1}\| \leq \frac{\nu+\nu_m-|\nu-\nu_m|}{4}\|\nabla w_h^{n+1}\|^2$$
$$+\frac{(\nu-\nu_m)^2}{4(\nu+\nu_m-|\nu-\nu_m|)}\|\nabla(v_h^{n+1} - 2v_h^n + v_h^{n-1})\|^2.$$

Combining, we now have that

$$\frac{1}{4\Delta t}\big(\|v_h^{n+1}\|^2 - \|v_h^n\|^2 + \|2v_h^{n+1} - v_h^n\|^2 - \|2v_h^n - v_h^{n-1}\|^2 + \|w_h^{n+1}\|^2 - \|w_h^n\|^2 + \|2w_h^{n+1} - w_h^n\|^2$$
$$-\|2w_h^n - w_h^{n-1}\|^2 + \|v_h^{n+1} - 2v_h^n + v_h^{n-1}\|^2 + \|w_h^{n+1} - 2w_h^n + w_h^{n-1}\|^2\big) + \frac{\nu+\nu_m-|\nu-\nu_m|}{8}\big(\|\nabla v_h^{n+1}\|^2$$
$$+\|\nabla w_h^{n+1}\|^2\big) \leq \frac{(\nu-\nu_m)^2}{4(\nu+\nu_m-|\nu-\nu_m|)}\big(\|\nabla(v_h^{n+1} - 2v_h^n + v_h^{n-1})\|^2 + \|\nabla(w_h^{n+1} - 2w_h^n + w_h^{n-1})\|^2\big)$$
$$+\frac{2}{\nu+\nu_m-|\nu-\nu_m|}\big(\|f_1^{n+1}\|_{-1}^2 + \|f_2^{n+1}\|_{-1}^2\big). \quad (7.6)$$

The inverse inequality provides the estimate

$$\|\nabla(z_h^{n+1} - 2z_h^n + z_h^{n-1})\|^2 \leq C_i h^{-2} \|z_h^{n+1} - 2z_h^n + z_h^{n-1}\|^2,$$

which allows equation (7.6) to be written as

$$\begin{aligned}
&\frac{1}{4\Delta t}\big(\|v_h^{n+1}\|^2 - \|v_h^n\|^2 + \|2v_h^{n+1} - v_h^n\|^2 - \|2v_h^n - v_h^{n-1}\|^2 + \|w_h^{n+1}\|^2 - \|w_h^n\|^2 + \|2w_h^{n+1} - w_h^n\|^2 \\
&\qquad - \|2w_h^n - w_h^{n-1}\|^2\big) + \left[\frac{1}{4\Delta t} - \frac{(\nu-\nu_m)^2 C_i h^{-2}}{4(\nu+\nu_m-|\nu-\nu_m|)}\right] \|w_h^{n+1} - 2w_h^n + w_h^{n-1}\|^2 \\
&+ \left[\frac{1}{4\Delta t} - \frac{(\nu-\nu_m)^2 C_i h^{-2}}{4(\nu+\nu_m-|\nu-\nu_m|)}\right] \|v_h^{n+1} - 2v_h^n + v_h^{n-1}\|^2 + \frac{\nu+\nu_m-|\nu-\nu_m|}{8}\big(\|\nabla v_h^{n+1}\|^2 + \|\nabla w_h^{n+1}\|^2\big) \\
&\qquad \leq \frac{2}{\nu+\nu_m-|\nu-\nu_m|}\left(\|f_1^{n+1}\|_{-1}^2 + \|f_2^{n+1}\|_{-1}^2\right). \qquad (7.7)
\end{aligned}$$

Now using the assumption on the time step size and applying standard techniques completes the proof.

□