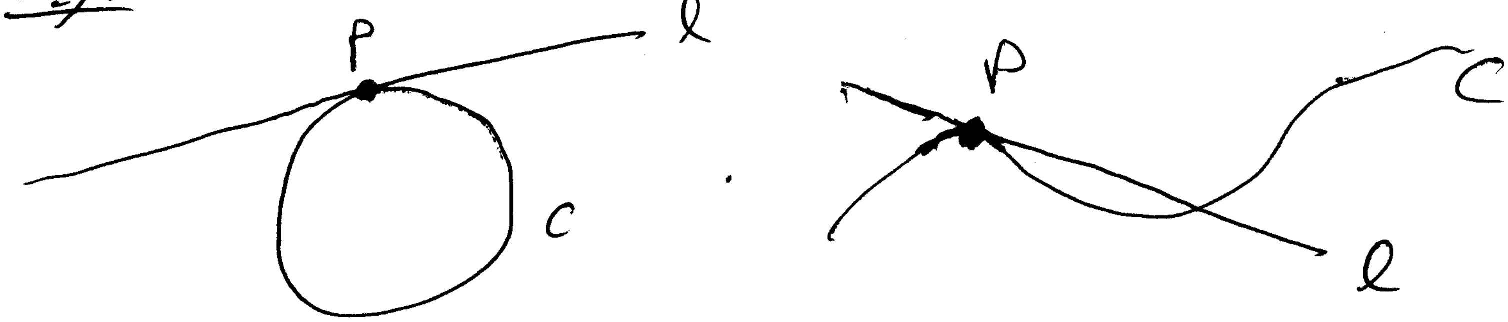
§2.1 Tangent & Velocity Problems.

Tangent Lines:

If C is a curve we say that a line l is tangent to C at a point P if l touches the curve C at P but doesnot pass through

e.g.

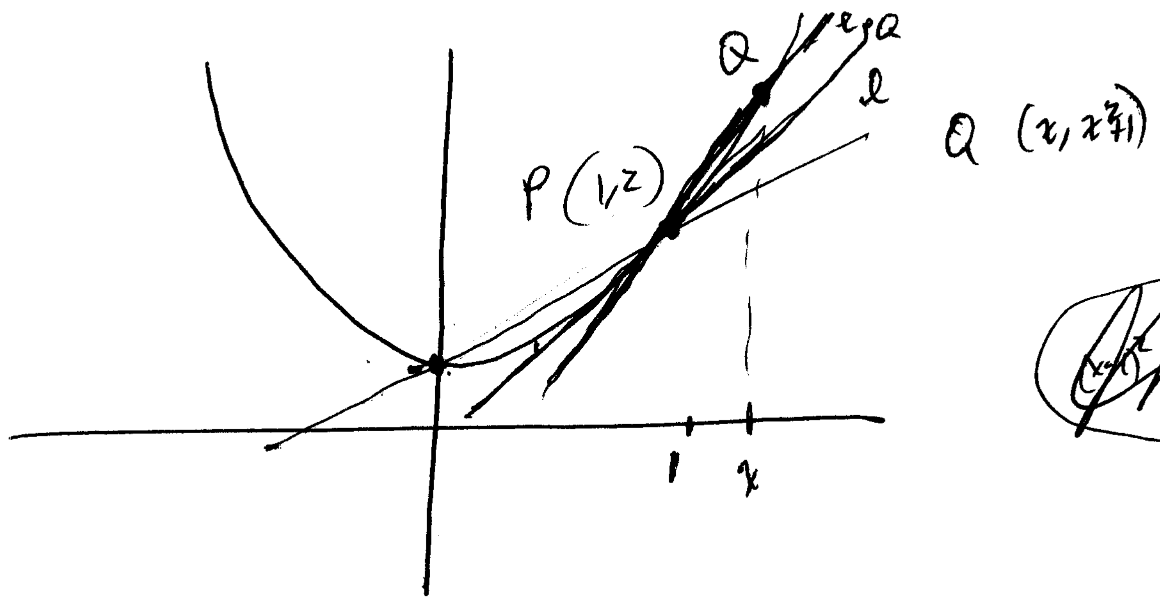


e.g.

Let C: $y = x^2 + 1$
Find the eq. of the ~~tangent~~ line tangent to C at $(1,2)$

Note: We know l passes through $(2,5)$ so, if we can determine the slope m_l of l we can write ~~the~~ eq. for l, namely

$$y - 2 = m_l(x - 1)$$



~~$2x = x^2+1$~~
~~$x^2 - 2x + 1 = 0$~~
~~$(x-1)^2 = 0$~~
~~$x = 1$~~

$$m_{PQ} = \frac{(x^2+1)-2}{x-1} = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{(x-1)} = x+1$$

Let's move Q closer to P
i.e. move x close to 1.

Right side

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

Left side

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

We say that the slope of the tangent lines is the limit of the slopes of the secant lines.

$$\lim_{Q \to P} m_{PQ} = m$$

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$$

Guess: $m = 2$

$\ell: y = 2x$

Velocity:

Ex a ball is dropped from a height of 450 m. Find the instantaneous velocity after 5 seconds.

Dist fallen after t seconds is

(27.449

$$s(t) = 4.9t^2$$

Recall:

$$\text{avg. velocity} = \frac{\text{dist. traveled}}{\text{time elapsed}}$$

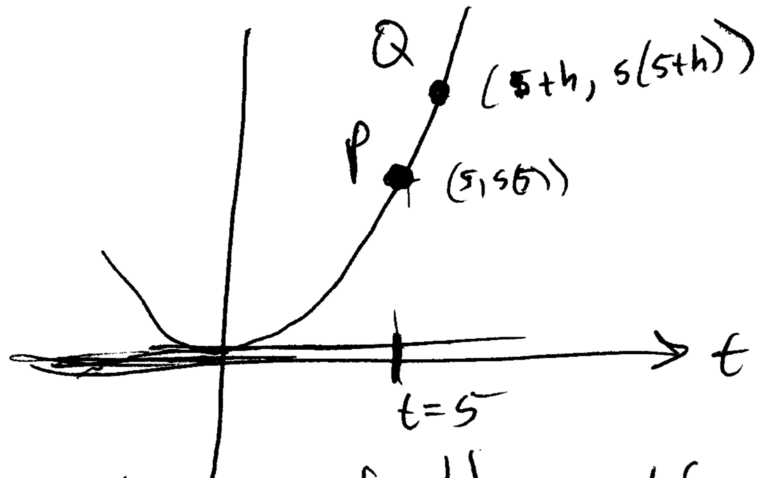
So, avg. vel. for time interval [5, 5.1]

$$\frac{s(5.1) - s(5)}{0.1} = 49.49 \text{ m/s}$$

Time interval	Avg. vel.
[5, 6]	53.9
[5, 5.1]	49.49
[5, 5.01]	49.245
[5, 5.001]	49.049

- Instantaneous velocity at $t=5$ appears to be $49\ m/s$

- Let $y = s(t) = 4.9t^2$



Find the slope of the line tan to $y = s(t)$ at $(5, s(5))$

$$m_{PQ} = \frac{s(5+h) - s(5)}{(5+h) - 5} = \frac{4.9(5+h)^2 - 4.9(5)^2}{h}$$

$$= \frac{4.9h^2 + 49h}{h} = 4.9h + 49$$

As $Q \longrightarrow P$, $h \longrightarrow 0$

Thus, $m_{PQ} \longrightarrow 49$

The slope of the tangent line is 49.

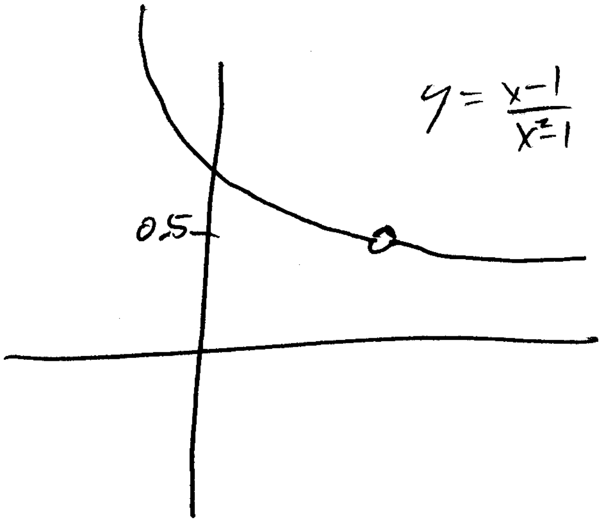
- Make up a poly. $f(x) = x^2 - x + 2$
- Investigate behavior near 2
  Plug in values near 2 but don't plug in 2.

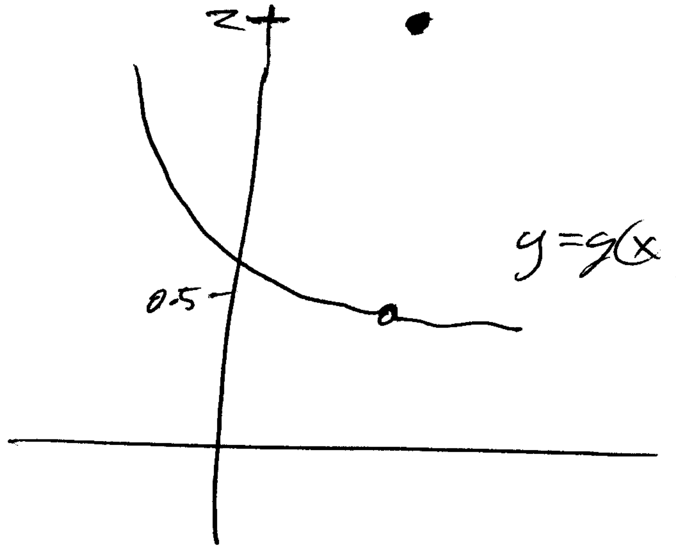
- $\lim_{x \to 2} (f(x)) =$ ______

Def: We write
$$\lim_{x \to a} f(x) = L$$
and say "the limit of $f(x)$ as x approaches a is L"
provided we can make the value of $f(x)$ ~~arbitrarily close to L~~ as close as we like to L by taking x suff. close to but not equal to a.

eg: (1) $\lim_{x \to 1} \frac{x-1}{x^2-1} =$?

(2) $g(x) = \begin{cases} \frac{x-1}{x^2-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$ $\quad \lim_{x \to 1} \left(\frac{x-1}{x^2-1}\right) =$?





e.g. **Guess!** (Groups)

$$\lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2} = \_\_\_\_ \qquad \frac{1}{6} = 0.16666$$

$$\lim_{x \to 0} \frac{\sin(x)}{x}$$

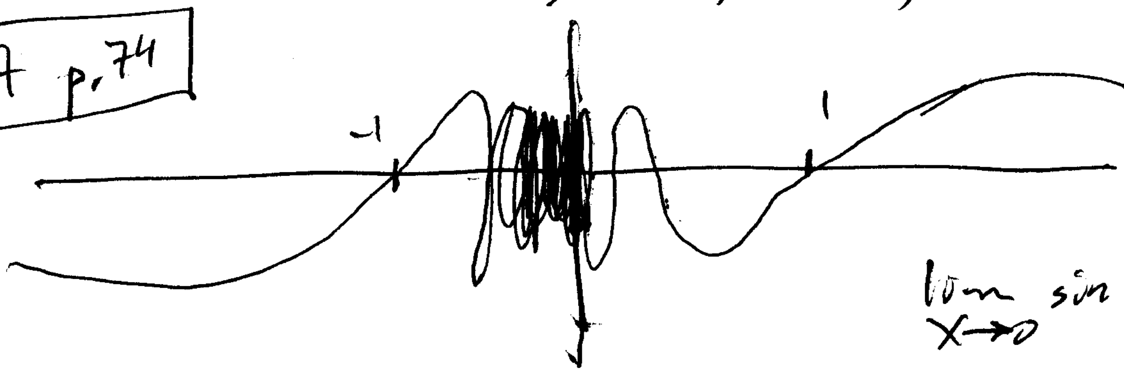
e.g. $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$

Try $x = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots$

Now try $x = \frac{2}{5}, \frac{2}{49}, \frac{2}{101}, \frac{2}{1001}$

$x = \frac{2}{3}, \frac{2}{51}, \frac{2}{103}, \frac{2}{1003}$

$\boxed{\text{Fig. 7 p. 74}}$



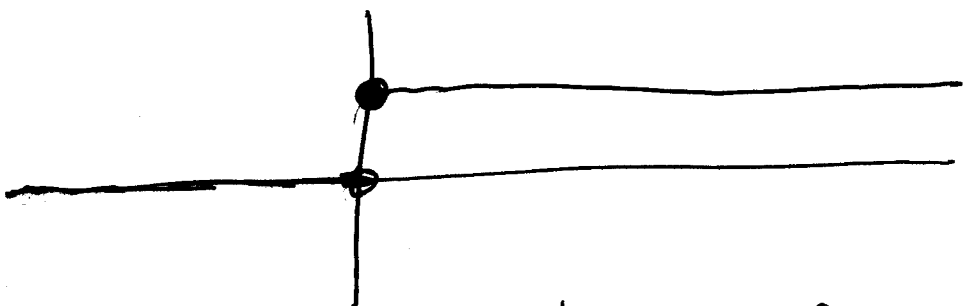
$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$ does not exist!

Note: We will need more mathematical tools to calculate limits.

e.g. $$\lim_{x\to 1}\left(\frac{x-1}{x^2-1}\right) = \lim_{x\to 1}\left[\frac{x-1}{(x+1)(x-1)}\right] \overset{\text{if } x\neq 1}{=} \lim_{x\to 1}\left[\frac{1}{x+1}\right]$$
$$= \frac{1}{2}$$

e.g.: Heavy side fc.

$$H(t) = \begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t\geq 0 \end{cases}$$



what is the limit as $t \to 0$ of $H(t)$?

Def: We write
$$\lim_{x\to a^-} f(x) = L$$
and say the left-hand limit of $f(x)$ as x approaches a from the left is L if we can make $f(x)$ as close to L as we like by taking x suff. close to a and $x<a$.

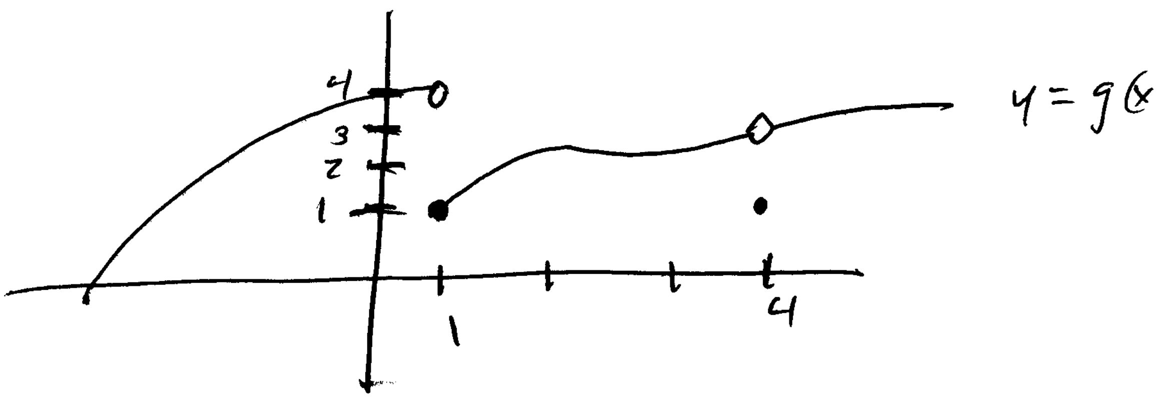
Def: Right hand limit

$$\lim_{x \to a^+} f(x) = L$$

Note:

$$\lim_{x \to a} f(x) = L \quad \underline{\text{iff}} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$

eg.:



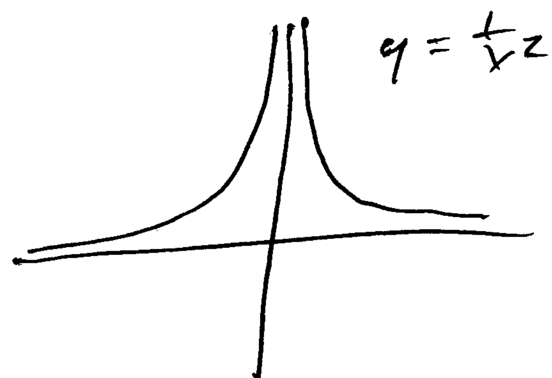
$\lim_{x \to 1^-} g(x) =$ $\qquad$ $\lim_{x \to 1^+} g(x) =$ $\qquad$ $\lim_{x \to 1} g(x) =$

$\lim_{x \to 4^-} g(x) =$ $\qquad$ $\lim_{x \to 4^+} g(x) =$ $\qquad$ $\lim_{x \to 4} g(x) =$

Infinite Limits:

eg.: $\lim_{x \to 0} \frac{1}{x^2}$

x	$1/x^2$
± 1	1
± 10	0.01
± 100	$\pm .0001$



$\lim_{x \to 0} \frac{1}{x^2} = \infty.$

Note: $\infty \notin \mathbb{R}$.

- **Def:** Let f be a fcn defined on both sides of a.

$$\lim_{x\to a} f(x) = \infty$$

provided $f(x)$ can be made as large as we like by taking x suff. close to but not equal to a.

Def: Similarly we define

$$\lim_{x\to a} f(x) = -\infty$$

eg.:
$$\lim_{x\to 0} \frac{-1}{x^2} = -\infty.$$

- **Def:** The line $x = a$ is a vert. asymptote provided one of the following is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$

$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty.$$

eg.:
$$\lim_{x\to 3^-} \frac{2x}{x-3}, \qquad \lim_{x\to 3^+} \frac{2x}{x-3}$$