Precise Def. of limit:

eg.: Compute

$$\lim_{x\to 1} [2x+3] = 2\lim_{x\to 1} x + 3$$

$$= 5$$

So, this means that we can make

$$f(x) = 2x+3$$

as close to 5 as we like by taking x close to but not equal to 1.

O.K. Suppose we want

$$|f(x) - 5| < 0.01$$

How close does x have to be to 1?

$$|f(x) - 5| = |2x+3-5| = |2x-2| = 2|x-1|$$

So, we want

$$2|x-1| < 0.01$$

$$\Leftrightarrow \quad |x-1| < 0.005$$

That is, $0.995 < x < 1.005$

As long as $|x-1| < 0.005$, $|f(x) - 5| < 0.01$

What if we choose a smaller bound than 0.01?
Let's say we want

$$|f(x) - 5| < \varepsilon$$

where $\varepsilon > 0$ is some small bound.

As before, we compute

$$|f(x) - 5| = 2|x-1|$$

Thus

$$|f(x) - 5| < \varepsilon$$

$$\iff 2|x-1| < \varepsilon$$

$$\iff |x-1| < \varepsilon/2$$

Put $\delta = \varepsilon/2$.

Then we can say that whenever $\underbrace{|x-1| < \delta}_{\text{when } x \text{ is within } \delta \text{ of } 1}$ then $\underbrace{|f(x)-5| < \varepsilon}_{f(x) \text{ is within } \varepsilon \text{ of } 5.}$

This is what we mean when we say

$$\lim_{x \to 1} (f(x)) = 5.$$

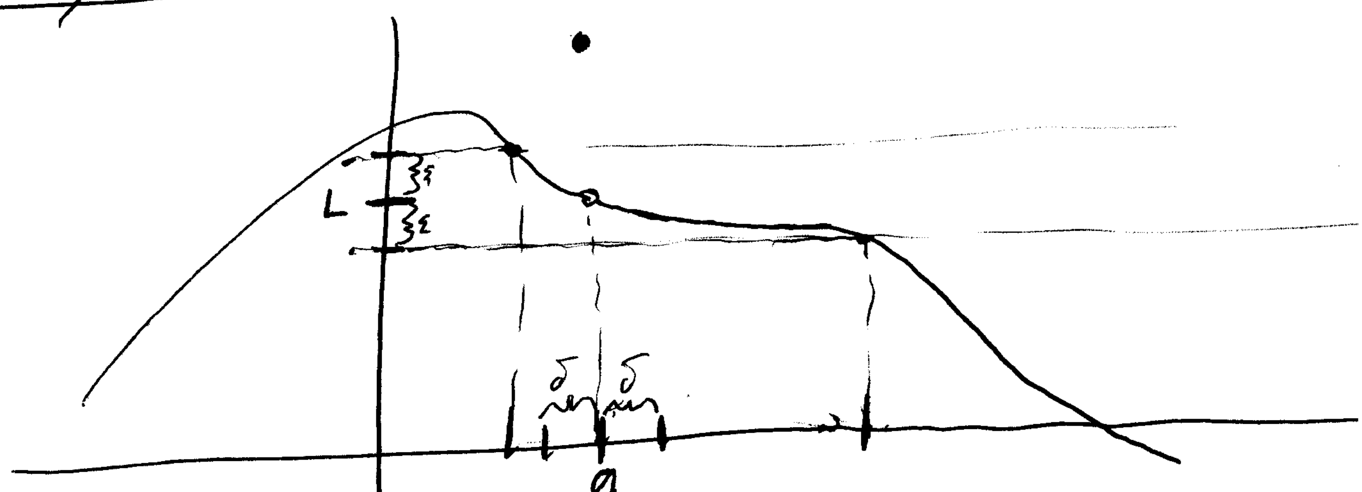
• **Def:** Let f be a fc. defined on some open interval that contains a except possibly at a itself. Then we write

$$\lim_{x \to a} [f(x)] = L$$

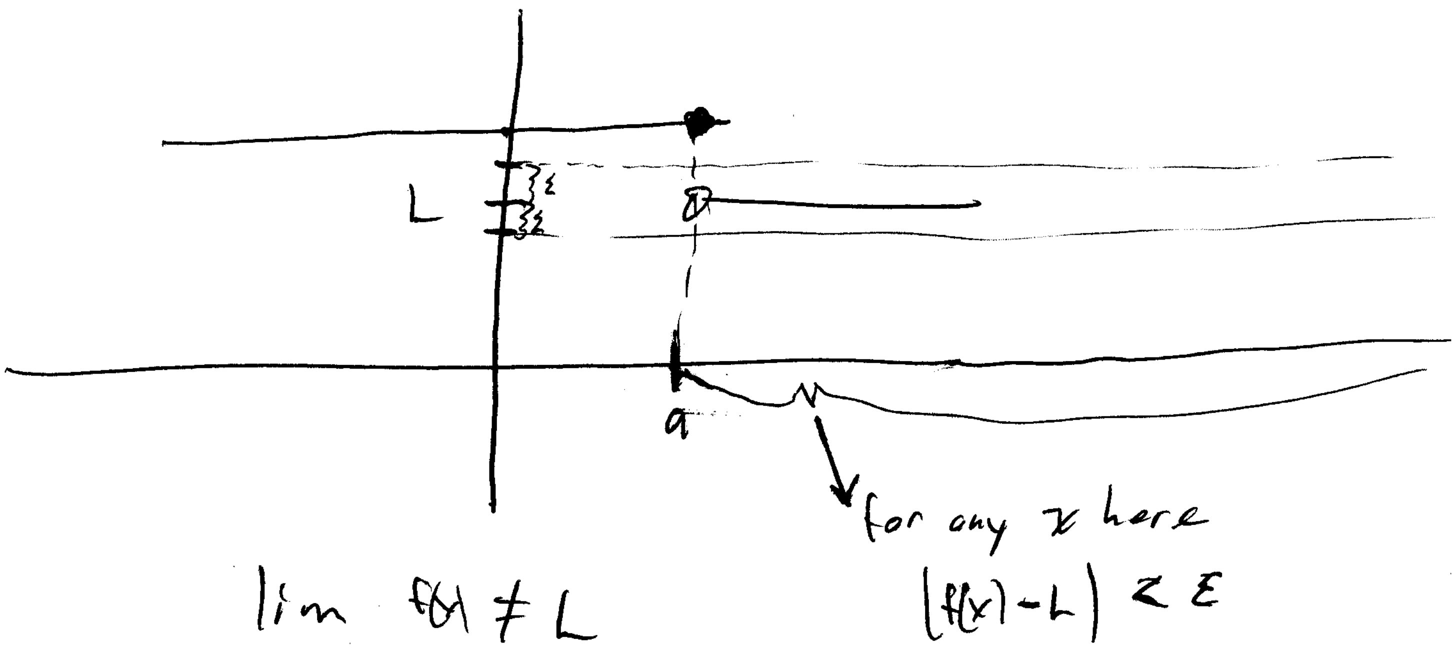
provided that for any $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $|x-a| < \delta$, $|f(x) - L| < \varepsilon$.

$$\left(|x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon \right).$$

Graphically:



$$\boxed{\lim_{x \to a} f(x) = L}$$



$\lim_{x \to a} f(x) \neq L$

For any $x < a$, $|f(x) - L| > \varepsilon$. Thus

$\lim_{x \to a} f(x) \neq L$.

e.g., Prove that $\lim_{x \to 4} (3x - 7) = 5$.

1.) Find δ.

2.) Prove it works.

- **Def:** (Left Hand limit)

$$\lim_{x \to a^-} f(x) = L$$

provided that for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$|f(x) - L| < \varepsilon$$

whenever $a - \delta < x < a$.

- **Def:** (Right hand limit)

$$\lim_{x \to a^+} f(x) = L$$

provided that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever

$$a < x < a + \delta.$$

- **eg.** Prove that

$$\lim_{x \to 0^+} (\sqrt{x}) = 0$$

1.) Guess δ.

2.) Prove it works.

Infinite Limits.

Def: $$\lim_{x \to a} f(x) = \infty$$

provided that for every $M > 0$ there is $\delta > 0$ such that

$$f(x) > M \quad \text{whenever} \quad |x - a| < \delta$$

(i.e. $|x-a| < \delta \implies f(x) > M$)

Note: You should think of M as being very large.

e.g. Prove $$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

1.) Guess δ.

$$\delta = \frac{1}{\sqrt{M}}$$

2.) Prove it works

$$|x - 0| = |x| < \frac{1}{\sqrt{M}}$$

$$\Rightarrow x^2 < \frac{1}{M}$$

$$\Rightarrow M < \frac{1}{x^2}$$

$$\Rightarrow \frac{1}{x^2} > \frac{1}{M}.$$

Def: $$\lim_{x \to a} f(x) = -\infty$$

provided that for every negative # N there is $\delta > 0$ such that $|x-a| < \delta \implies f(x) < N$.