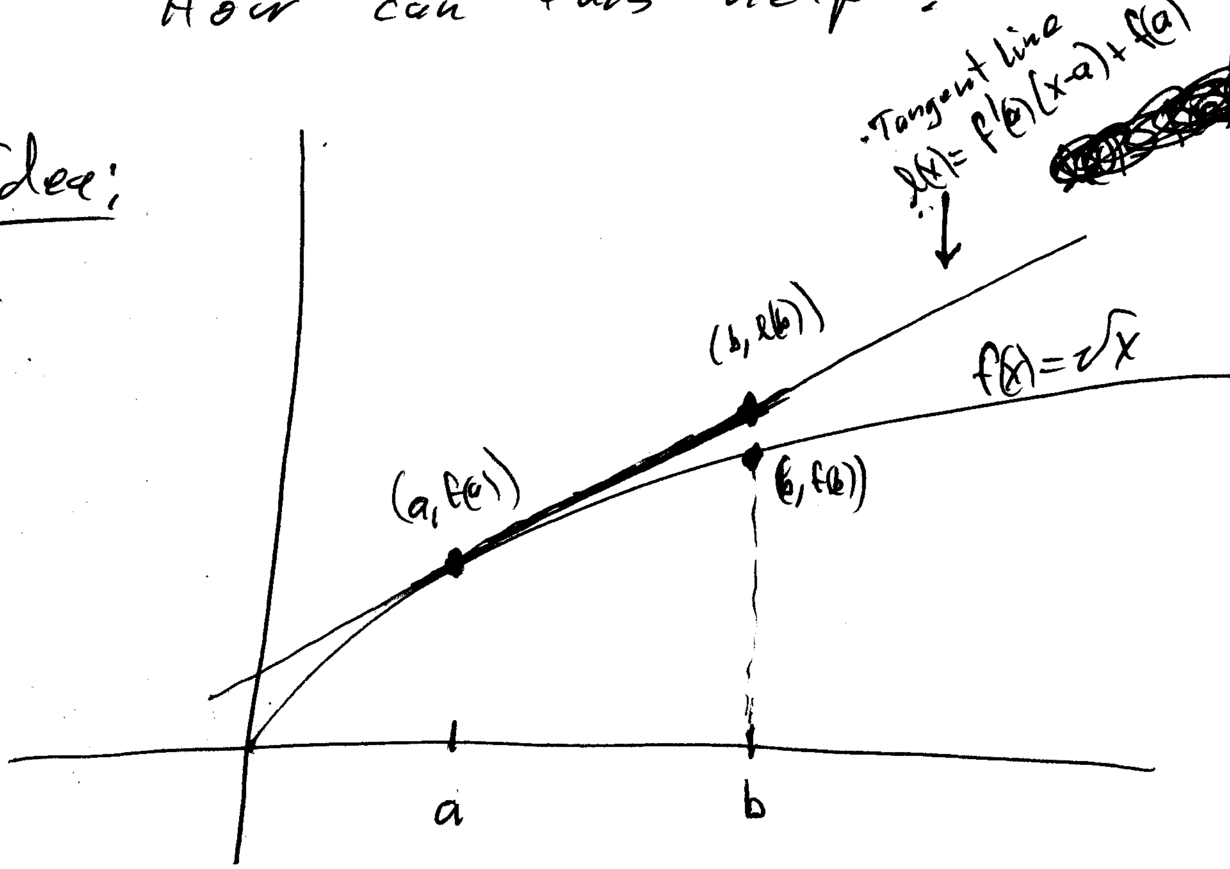
§ 3.10 Linear Approximation & Differentials.

example: Approximate $\sqrt{5}$.

- **Note:** We know $\sqrt{4} = 2$.
  How can this help?

- **Idea:**



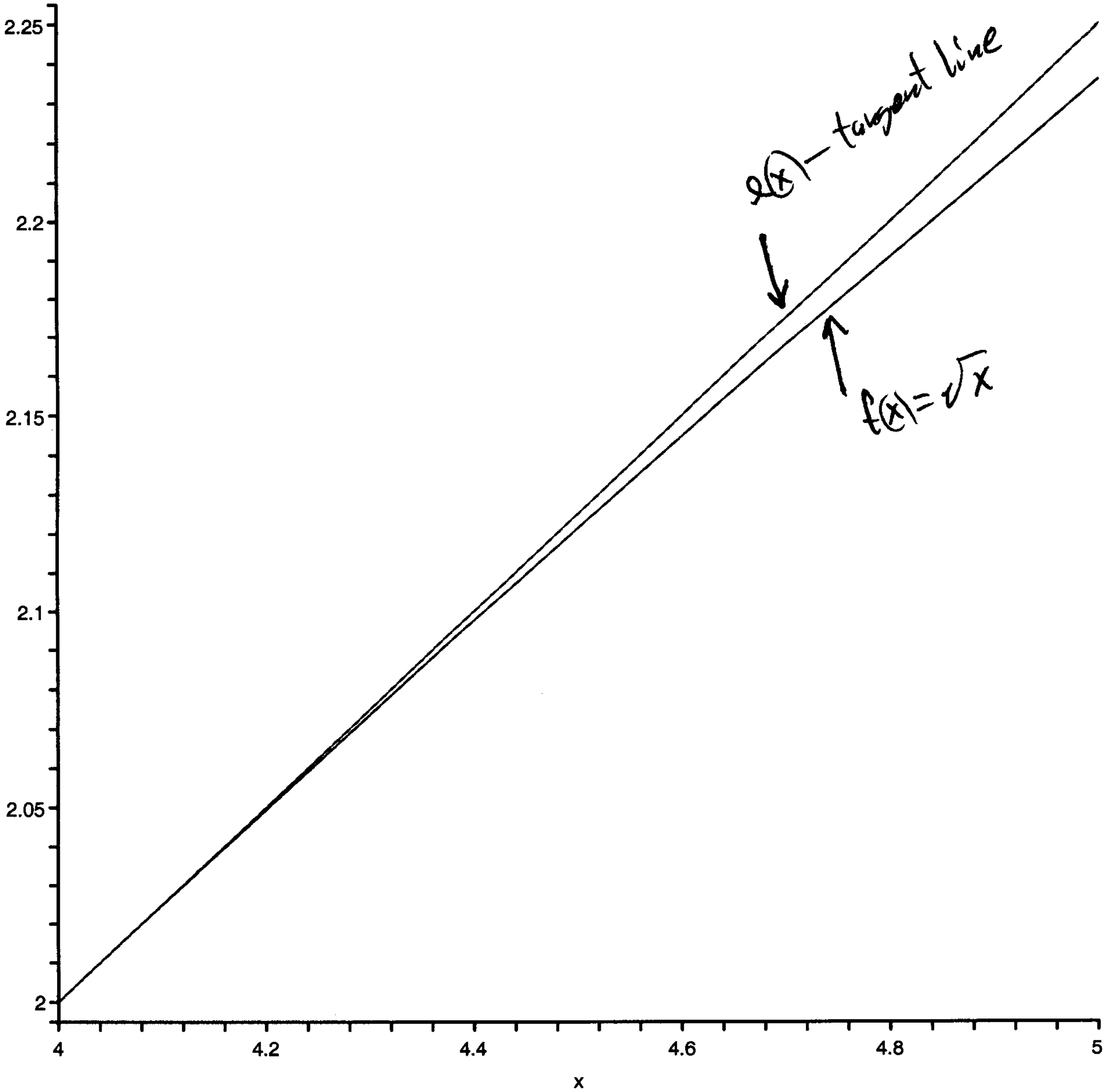
Note: If b is close to a then $l(b)$ is close to $f(b)$

Thus,

$$f(b) \approx l(b) = f'(a)(b-a) + f(a)$$

Now take $f(x) = \sqrt{x}$

$\Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$



Note that on the above graph it appears that $l(5)$ is ver close to $f(5)$ (within 0.02).

So, to approximate $\sqrt{5}$ we use the tangent line to $f(x) = \sqrt{x}$ at $(4,2)$.

$$\sqrt{5} = f(5) \approx \ell(5) = f'(4)(x-4) + f(4)$$

Note: $f(4) = \sqrt{4} = 2$

$$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

So ~~$f(5) \approx f'(4)(x-4) + f(4)$~~

$$\sqrt{5} \approx \frac{1}{4}(5-4) + 2 = 2.25.$$

(Your calculator might give $\sqrt{5} \approx 2.236067977$)

Def: We call the function

$$L(x) = f(a) + f'(a)(x-a)$$

the linearization of f at a.

e.g. Compute the linearization of
$f(x) = \sqrt{x+6}$ at $a=3$

Compare the values of $L(x)$ & $f(x)$ near 3

x	$L(x)$	$f(x)$
3		
2,9		
3,1		
2,5		
3,5		
2,2		
3,7		
2		
4		

eg: For what values of x is the linear approx.

$$\sqrt{x+6} \approx 3 + \frac{1}{6}(x-3)$$

accurate to within 0.5?

Solve

$$\left| \sqrt{x+6} - \left(3 + \tfrac{1}{6}(x-3)\right) \right| < 0.5$$

$$\left| \sqrt{x+6} - \left(\tfrac{1}{6}x + \tfrac{5}{2}\right) \right| < 0.5$$

OR equivalently

$$\tfrac{1}{6}x + \tfrac{5}{2} - .5 < \sqrt{x+6} < \left(\tfrac{1}{6}x + \tfrac{5}{2}\right) + .5$$

$$\tfrac{1}{6}x + 2 < \sqrt{x+6} < \tfrac{1}{6}x + 3$$

$$\tfrac{1}{36}x^2 + \tfrac{2}{3}x + 4 < x+6 < \tfrac{1}{36}x^2 + x + 9$$

$$\tfrac{1}{36}x^2 - \tfrac{1}{3}x - 2 < 0 \quad \& \quad 0 < \tfrac{1}{36}x^2 + 3$$

$$x^2 - 12x - 72 < 0 \qquad 0 < x^2 + 108$$

always true

$$(x-6)^2 - 36 - 72 < 0$$

$$(x-6)^2 < 108$$

$$|x-6| < \sqrt{108}$$

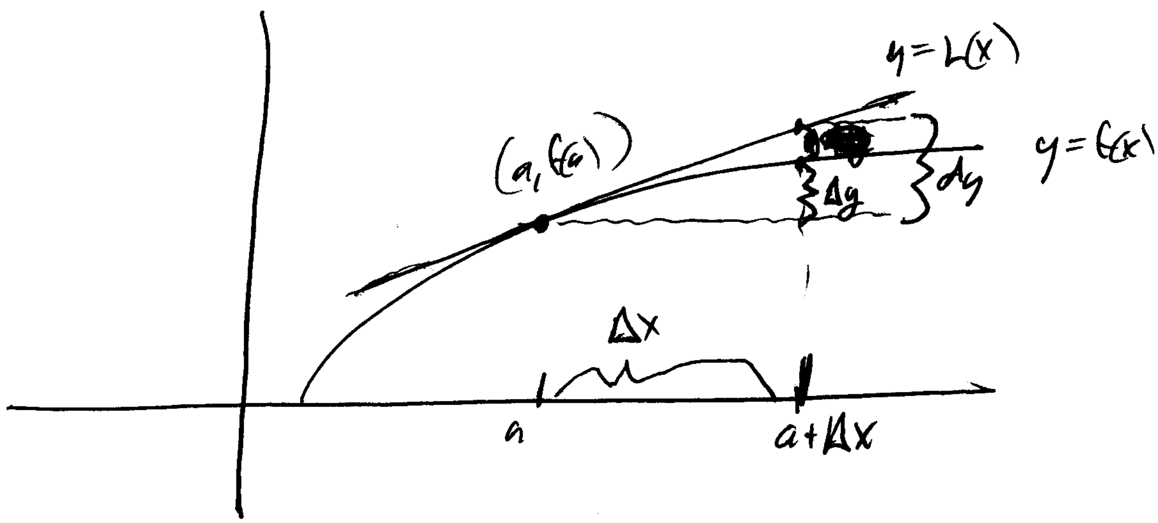
$$6 - \sqrt{108} < x < 6 + \sqrt{108}$$

See p. 207 for a geometric interpretation,

Differentials:

Def: The differential dx is an independent variable (like x) and we define dy as

$$dy = f'(x)\,dx$$



So, $\Delta y \approx dy$

Note: dy depends on the values of x and dx.

<u>eg.:</u> The radius of a sphere is measured ~~to~~ to be 21 cm with a possible error of at most 0.05 cm. What is the maximum error in ~~the~~ using this measure for the radius to compute the volume.