§4.2 The mean value theorem.

- **Rolle's Theorem:**

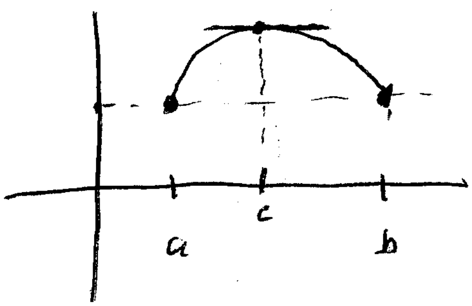
  Let f be a function satisfying:

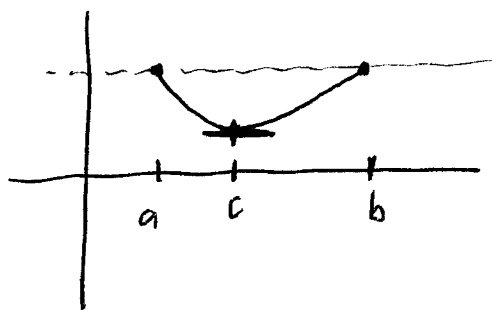
  1.) f is cont. on $[a,b]$.

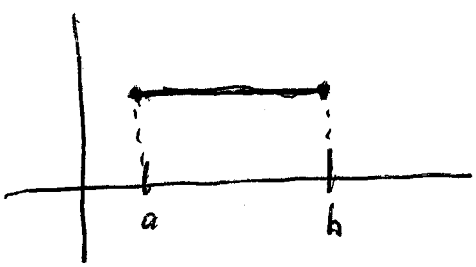
  2.) f is diff. on (a,b).

  3.) $f(a) = f(b)$

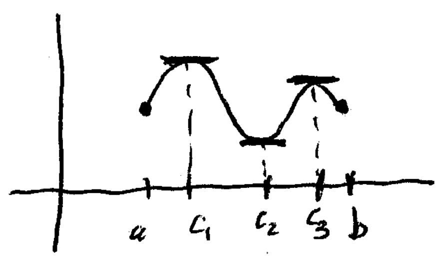
  Then there is a number $c \in (a,b)$ such that $f'(c) = 0$.







$f'(c)=0 \;\forall\, c \in (a,b)$



e.g.: Velocity.

If an object is in the same position at time $t=a$ and time $t=b$ and its position fcn is cont. and diff. then there is some ~~point~~ time $t=c$ $(a<b<c)$ such that the velocity at $t=c$ is 0.

e.g.: Show that

$$f(x) = 4x^3 + 2x - 16$$

has exactly one real root.

(1.) Note that $f(0) = -16$ and $f(2) = 20$ since f is cont. and since

$$-16 < 0 < 20,$$

the Intermediate Value Theorem says that there exists a number c with $0 < c < 2$ and $f(c) = 0$. So there is at least one root.

(2.) ~~If there were two roots c and d~~

~~Then 1.) f is cont. on [c,d]~~

~~2.) f is diff. on (c,d)~~

~~3.) f(c) = 0 = f(d)~~

~~Thus Rolle's thm would imply there is a number e with c < e < d and f'(e) = 0.~~

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(2) Suppose that there are 2 roots, say c and d. Then we have

1.) f is cont. on $[c,d]$

2.) f is diff on (c,d)

(In fact, $f'(x) = 12x^2 + 2$ exists on $(-\infty, \infty)$.)

3.) $f(c) = 0 = f(d)$

Thus, Rolle's Thm. implies that there is a number e with $c < e < d$ and $f'(c) = 0$

However,

$$f'(c) = 12c^2 + 2 \geq 2 > 0$$

So, there cannot be a second root.

~~Quiz~~ # 10.

1.) Identify all local ~~and~~ and global extrema of

$$f(x) = x^3 - 9x^2 + 15x + 2$$

on $[0, 6]$.

2.) Prove that

$$g(x) = x^5 + 7x^3 + 2x + 7$$

has exactly one real root.

- Mean Value Theorem:

Let f be a function satisfying:

1.) f is cont. on $[a,b]$.

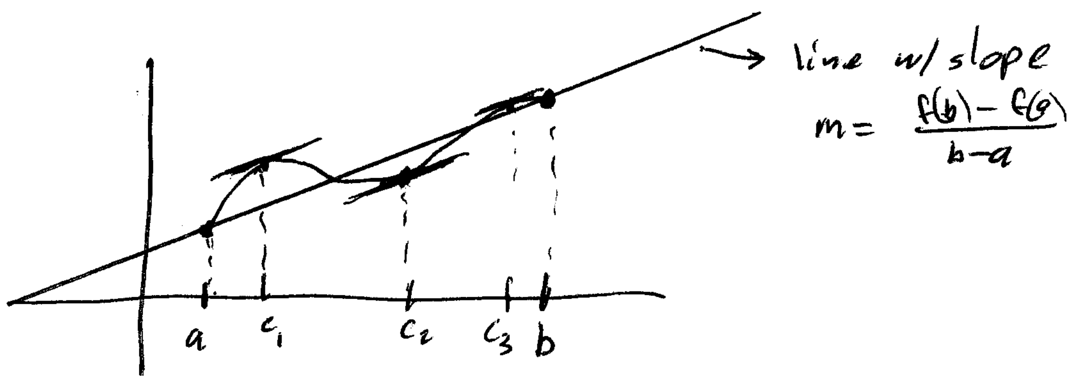
2.) f is diff. on (a,b)

Then there is a number $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

OR

$$f(b)-f(a) = f'(c)(b-a)$$



- eg. Consider the function

$$f(x) = x^3 - x$$

on the interval $[0,2]$.

Find $c \in (0,2)$ s.t. $f'(c) = \frac{f(2)-f(0)}{2-0}$.

<u>eg.:</u> What does MVT say about velocity?

Suppose that a car leaves a house and travels along a road and that its position on the road measured in miles from home is given by a cont. diff. fc. $s(t)$.
S also that

$$s(30)=20 \quad \& \quad s(90)=87$$

Was the driver speeding if the speed limit is 55 mph.?

M.V.T. in Layperson's terms:

If f is cont. and diff on $[a,b]$ then ~~there~~ at some point the instantaneous rate of change equals the average rate of change.

eg: S'pose $f(0) = 6$ and $f'(x) \leq 2$ for all values of x. How large ~~could~~ can $f(3)$ possibly be?

• Thm: If $f'(c) = 0$ for all $c \in (a,b)$ then f is constant on (a,b).

Pf: S'pose x_1 & $x_2 \in (a,b)$.

Then f is diff. on (x_1, x_2) and cont. on $[x_1, x_2]$.

Thus $\exists$ $c \in (x_1, x_2)$ s.t.

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

But $f'(c) = 0$ $\forall$ $c \in (a,b)$

$$\Rightarrow \quad f(x_2) - f(x_1) = 0.$$

Cor: If $f'(x) = g'(x)$ $\forall x \in (a,b)$ then $f(x) - g(x)$ is constant.

Thus, $f(x) = g(x) + C$ for some number C.