§4.4 Limits at infinity.

eg: Graph

$$f(x) = \frac{x^2-1}{x^2+1}$$

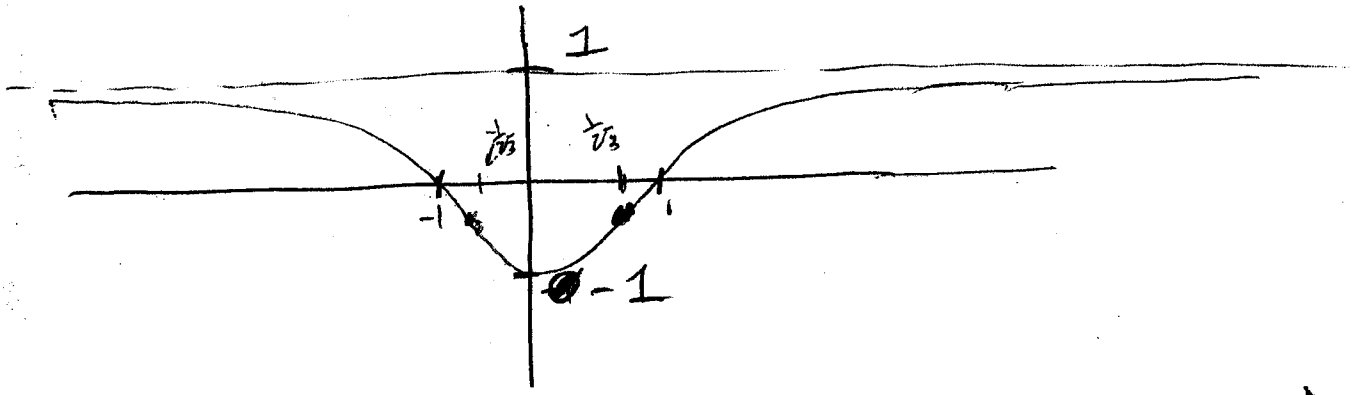
$f(x)=0 \iff x=\pm 1$
always defined & cont

$$f'(x) = \frac{4x}{(x^2+1)^2}$$

$f'(x)=0 \iff x=0$
f' always defined and co

$$f''(x) = \frac{4(1-3x^2)}{(x^2+1)^3}$$

$f''=0 \iff x=\frac{\pm 1}{\sqrt{3}}$
f'' always defined & cont.



(Note: $\frac{x^2-1}{x^2+1} \leq \frac{x^2-1}{x^2+1} + \frac{2}{x^2+1} = \frac{x^2+1}{x^2+1} = 1$)

Def: Suppose f is defined on (a, ∞).
Then

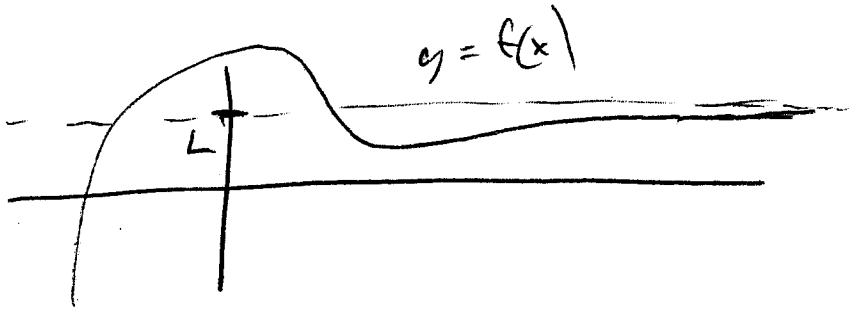
$$\lim_{x\to\infty} f(x) = L$$

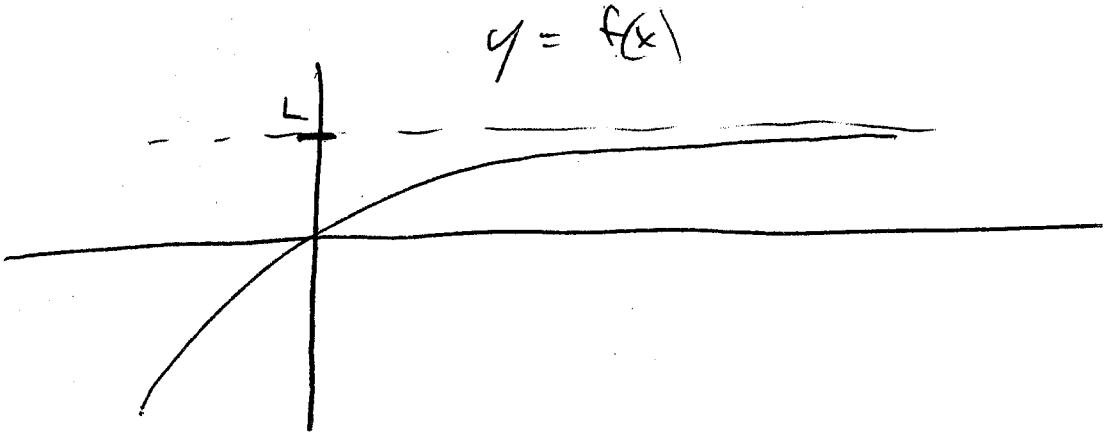
means that $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

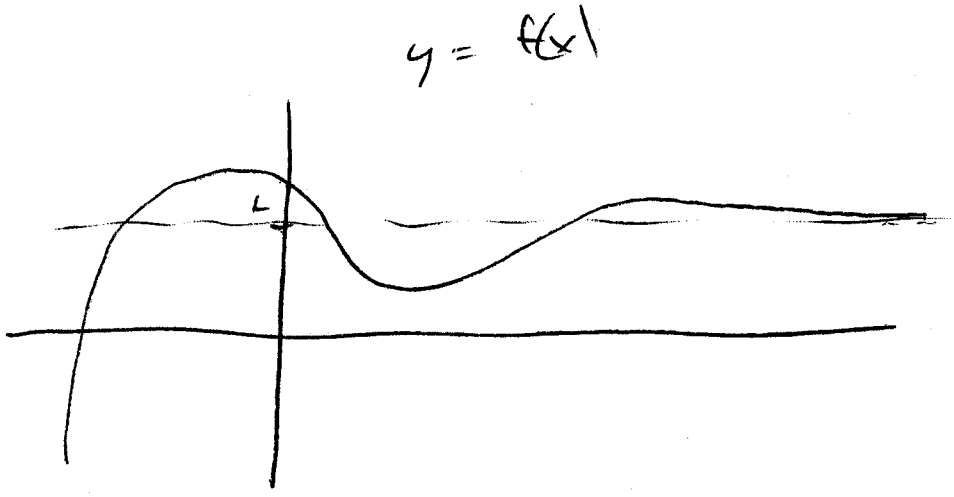
(Alt: $f(x) \to L$ as $x \to \infty$.)

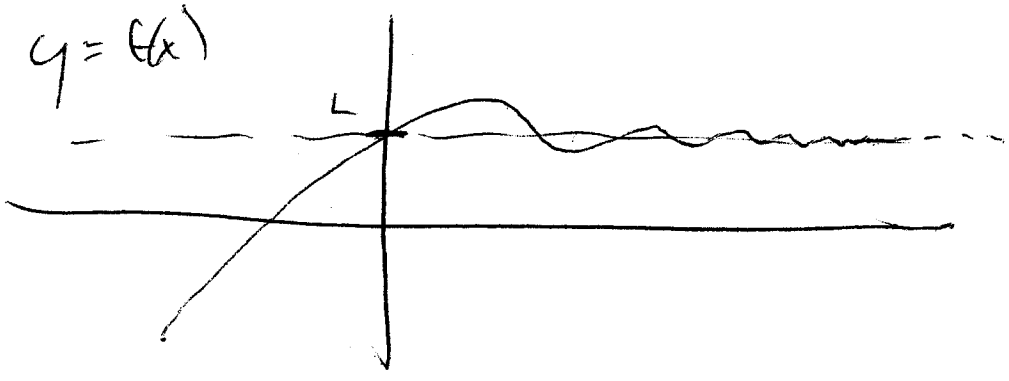
In all of the following pictures

$$\lim_{x \to \infty} f(x) = L$$









• **Def:** Let f be defined on $(-\infty, a)$

Then,

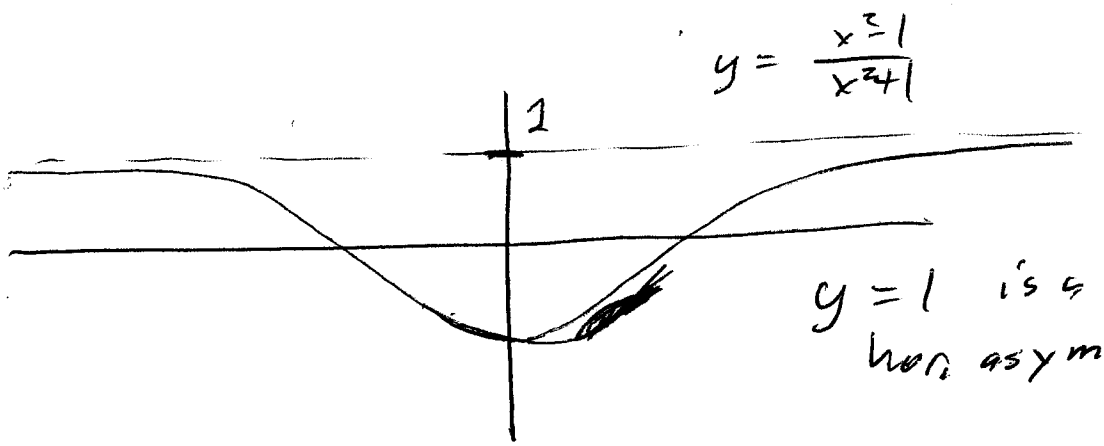
$$\lim_{x \to -\infty} f(x) = L$$

means that f can be made arbitrarily close to L by taking x sufficiently large & negative.

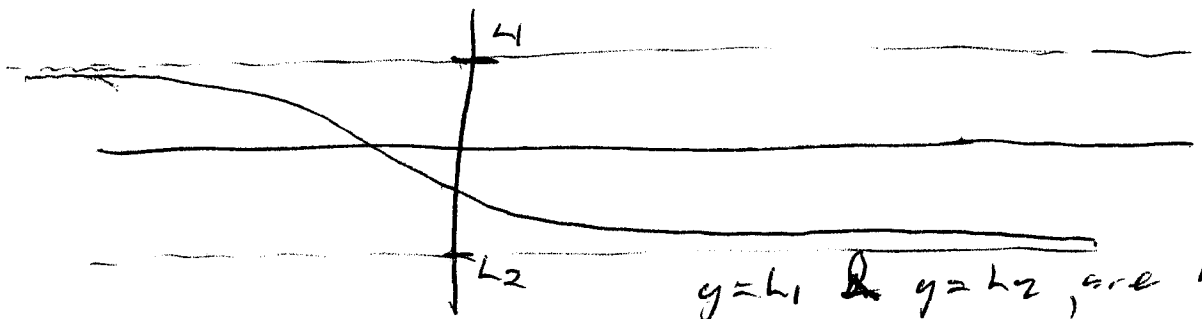
• **Def:** The line $y = L$ is a horizontal asymptote for $y = f(x)$ provided that either

$$\lim_{x \to \infty} f(x) = L \quad \underline{\text{OR}} \quad \lim_{x \to -\infty} f(x) = L$$

eg:



$y = 1$ is a hor. asym.



$y = L_1$ & $y = L_2$ are both hor. asym.'s.

eg: Compute $\lim_{x\to\infty} \frac{1}{x^2}$ & $\lim_{x\to-\infty} \frac{1}{x^2}$

Note: As x gets large, $\frac{1}{x^2}$ gets very small

In fact, for any $\varepsilon>0$, if we choose $x > \sqrt{\frac{1}{\varepsilon}}$ then

$$x^2 > \frac{1}{\varepsilon}$$

$$\Rightarrow \frac{1}{x^2} < \varepsilon$$

Similarly if we choose $x < -\sqrt{\frac{1}{\varepsilon}}$

Then

$$x^2 > \frac{1}{\varepsilon}$$

$$\Rightarrow \frac{1}{x^2} < \varepsilon$$

Thus, $\lim_{x\to\infty} \frac{1}{x^2} = 0$

$\lim_{x\to-\infty} \frac{1}{x^2} = 0$,

- Thm: If $r>0$, $r\in\mathbb{Q}$ then

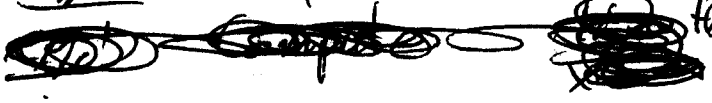
$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

In addition if x^r is defined $\forall x$ then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

eg. Compute:

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1},$$

eg. Let  $f(x) = \dfrac{\sqrt{5x^2 + 3}}{8x - 2}$,

Find the horizontal asymptotes.

Eg. Compute

$$\lim_{x\to\infty} \left(\sqrt{x^2+10} - x\right)$$

Eg: Evaluate $\lim_{x\to\infty} \sin\left(\frac{1}{x}\right)$

Eg: $\lim_{x\to\infty} \sin(x)$

Infinite Limits:

$\lim_{x \to \infty} f(x) = \infty$ $\quad$ $\lim_{x \to \infty} f(x) = -\infty$

$\lim_{x \to -\infty} f(x) = \infty$ $\quad$ $\lim_{x \to -\infty} f(x) = -\infty$

• Def:
$\lim_{x \to \infty} f(x) = \infty$ means that

$f(x)$ can be made arbitrarily large by taking x suff. large

eg: $\lim_{x \to \infty} x^3 = \infty$

($x > \sqrt[3]{M}$ $\Rightarrow x^3 > M$)

$\lim_{x \to -\infty} x^3 = -\infty$

($x < \sqrt[3]{-M}$ $\Rightarrow x^3 < -M$)

eg: $\lim_{x \to \infty} (x^2 - x)$

eg: $\lim_{x \to \infty} \dfrac{x^2 + x}{3 - x}$

eg: Use info. about the intercepts and [redacted] limits at ∞ & -∞ to sketch the graph of

$$f(x) = (x-2)^4(x+1)^3(x-1)$$

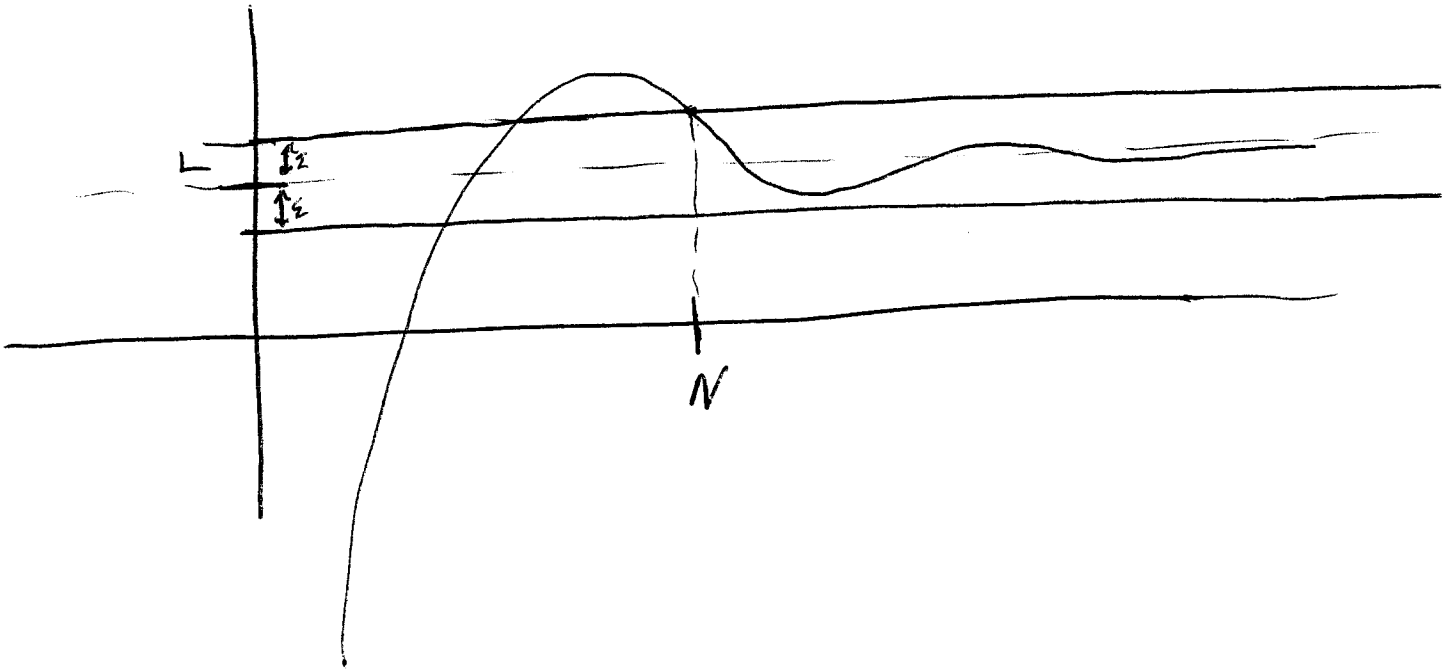
Precise Definitions:

- **Def:** Let f be a fc defined on (a, ∞). Then,

$$\lim_{x \to \infty} f(x) = L$$

means that for all $\varepsilon > 0$ there exists a number N such that

$$x > N \implies |f(x) - L| < \varepsilon$$



Ex. Prove: $\lim_{x \to \infty} \frac{1}{x} = 0$

Def: Let f be a fc on (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that $\forall N \geq 0$, $\exists M$ s.t.

$$x \geq M \Rightarrow f(x) > N.$$

4.5 Summary of Curve sketching.

To sketch the graph of $f(x)$:

A) Compute the domain of f.

B) Find the intercepts of f.

C) Symmetry. (even, odd, neither)

D) Asymptotes (Horizontal & Vertical)

E) Intervals of Inc./Dec. (use f' info)

F) Local Extrema

G) Concavity & Points of inflection (f'' info)

H) sketch.

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eg: sketch the graph of $y = \dfrac{5x^2}{x^2-1}$

eg.: sketch

$$y = \frac{x^2}{\sqrt{x+5}}$$

Slant Asymptotes:

Def: $y = mx+b$ is a "slant asymptote" of $y = f(x)$ provided that

$$\lim_{x \to \infty} \left(f(x) - (mx+b) \right) = 0$$

OR

$$\lim_{x \to -\infty} \left(f(x) - (mx+b) \right) = 0$$

eg. Graph $y = \frac{x^3}{x^2+1}$