§5.2 The definite integral.

Def: Let f be cont. on $[a,b]$.
Divide $[a,b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$

Let $a = x_0 < x_1 < x_2 < \dots < x_n = b$ be the endpoints of these intervals. Let $x_1^*, x_2^*, \dots, x_n^*$ be sample points from these intervals s.t. $x_i^* \in [x_{i-1}, x_i]$. Then we define the definite integral of f from a to b to be

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

• Note: If we take the sample points to be the right end points, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(a + i\Delta x)\,\Delta x$$

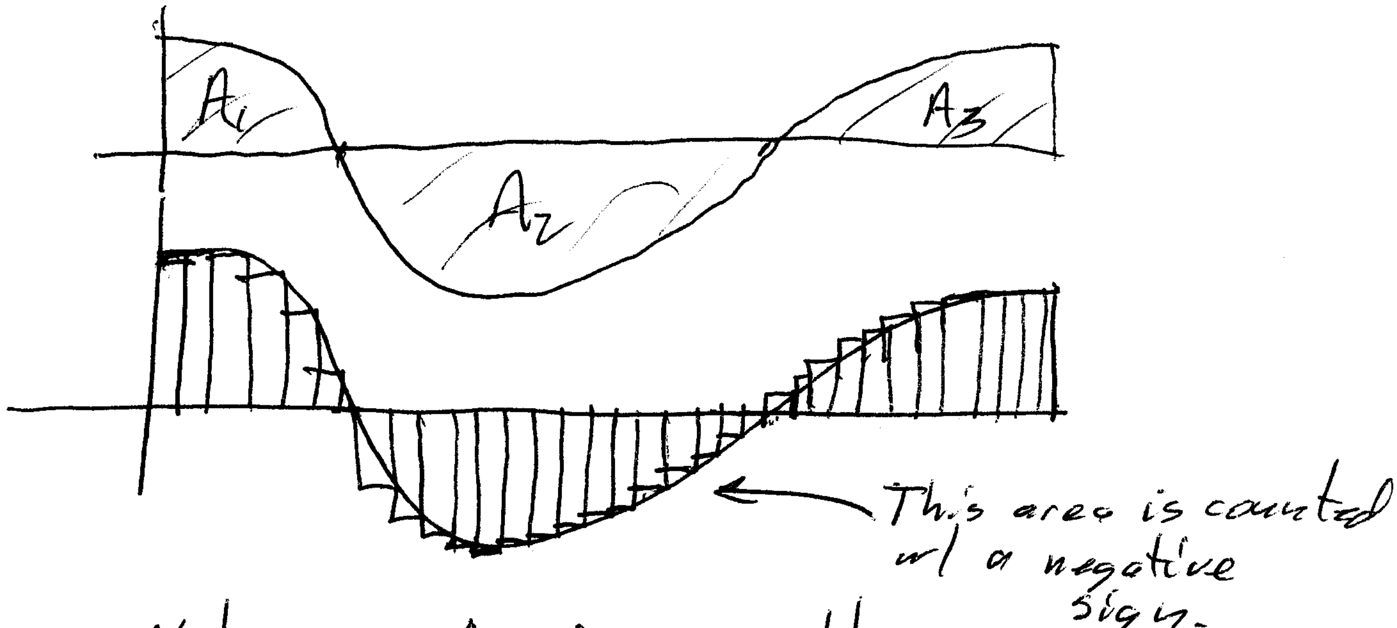
If we use left endpoints,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(a + (i-1)\Delta x)\,\Delta x$$

Note: $\int_a^b f(x)\,dx$ is a number and

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$$



* Note: If $f(x) < 0$ then $f(x_i^*)\Delta x < 0$.

Thus,

$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3$$

Precise Def:

$$\int_a^b f(x)\,dx = A$$

means for all $\varepsilon > 0$, $\exists$ an integer N such that if $n > N$ then

$$\left| A - \sum_{i=1}^{n} f(x_i^*)\Delta x \right| < \varepsilon$$

eg. Express $\lim_{n\to\infty} \sum_{i=1}^{n} \left(x_i^3 + x_i \cos(x_i) \right) \Delta x$ as an integral over $[0, \pi]$.

Summation identities.
$$\sum_{i=1}^{n} c = cn$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Properties:

$$\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} a_i + b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} a_i - b_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} (c a_i + d b_i) = c \sum_{i=1}^{n} a_i + d \sum_{i=1}^{n} b_i$$

__Eg:__ Evaluate

$$\int_0^3 (x^3 - 5x)\,dx \,.$$

Can this integral be interpreted in terms of area?

9. Evaluate.

$$\int_{2}^{5} x^4 \, dx$$

eg: Evaluate $\int_0^3 \sqrt{9-x^2}\,dx$ by interpreting it as an area.

- Mid Point Rule:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and

$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$$

eg: Use the mid point rule w/ $n=5$ to approximate $\int_1^5 \frac{1}{x}\,dx$

Properties of the Definite integral.

1.) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2.) $\int_a^a f(x)\,dx = 0$

3.) $\int_a^b c\,dx = c\,(b-a)$

4.) $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

5.) $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

6.) $\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_c^b g(x)\,dx$

eg.1. Compute $\int_0^1 (4+3x^2)\,dx$

Fact:

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

eg: If $\int_0^{10} f(x)\,dx = 20$ and $\int_0^8 f(x)\,dx = 10$

then

$$\int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx - \int_0^8 f(x)\,dx$$

$$= 20 - 10 = 10.$$

Comparison Properties:

1.) If $f(x) \geq 0$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \geq 0$$

2.) If $f(x) \geq g(x)$ $\forall$ $a \leq x \leq b$ then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

3.) If $m \leq f(x) \leq M$ $\forall$ $a \leq x \leq b$ then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

e.g.: Estimate $\int_1^4 \sqrt{x}\,dx$

e.g.: Estimate $\int_{10}^{12} \left(10 + \frac{1}{100}\sin(x)\right) dx$