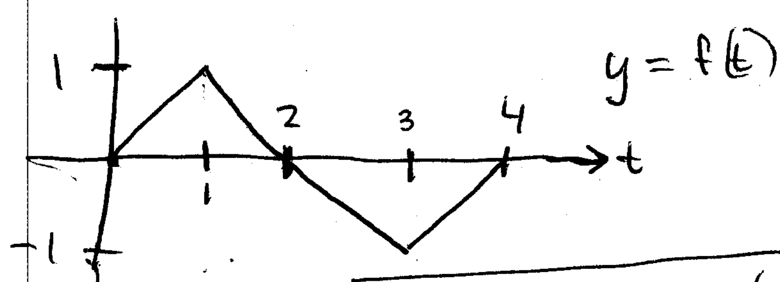
§5.3 The Fundamental Theorem of Calculus.

Accumulation functions:

Suppose that f is cont. on (a, ∞) then we define a function $g(x)$ as

$$g(x) = \int_a^x f(t)\,dt$$

• **eg.:**



Put $g(x) = \int_0^x f(t)\,dt$

$$g(0) = \int_0^0 f(t)\,dt = 0$$

$$g(1) = \int_0^1 f(t)\,dt = \tfrac{1}{2} \times 1 \times 1 = \tfrac{1}{2}$$

$$g(2) = \int_0^2 f(t)\,dt = \tfrac{1}{2}(2) \times (1) = 1$$

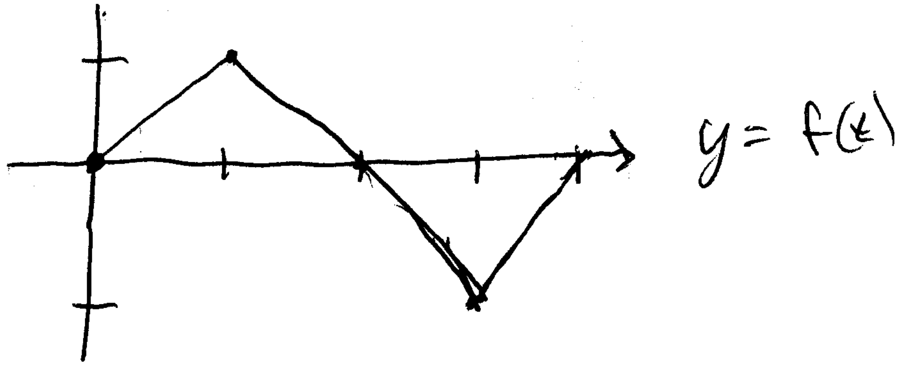
$$g(3) = \int_0^3 f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^3 f(t)\,dt$$

$$= 1 - \tfrac{1}{2}(1)(1) = \tfrac{1}{2}$$

$$g(4) = \int_0^4 f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^4 f(t)\,dt$$

$$= 1 - \tfrac{1}{2}(2)(1) = 0,$$

Sketch the graph of $g(x) = \int_0^x f(t)\,dt$

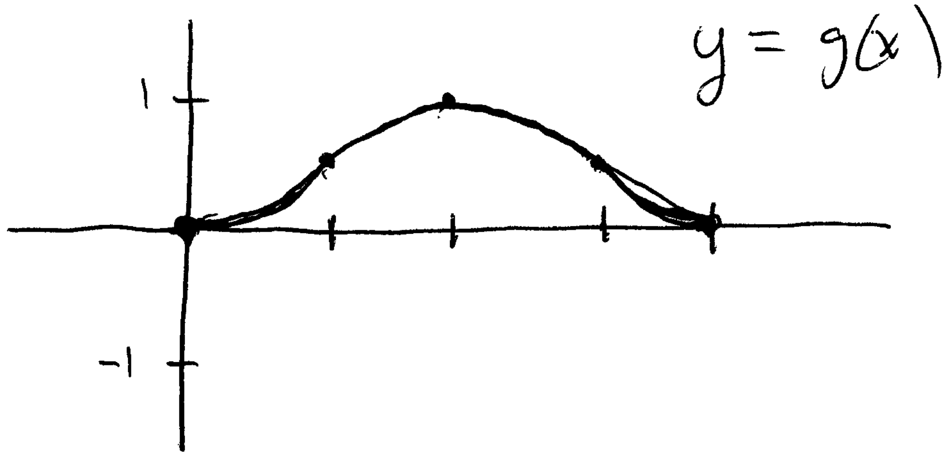


Note: g is increasing on ________

g is decreasing on ________

g attains a max at $x =$ ________

g attains a min at $x =$ ________



eg.:
Take

$$f(t) = t \quad \text{and } a = 0$$

Then

$$g(x) = \int_0^x f(t)\,dt$$

$\Delta t = \frac{x}{n}$

$t_i = 0 + \frac{xi}{n}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\Delta t$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{xi}{n} \cdot \frac{x}{n} \right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{x^2}{n^2}\, i$$

$$= \lim_{n\to\infty} \frac{x^2}{n^2} \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{x^2}{n^2} \cdot \frac{n(n+1)}{2}$$

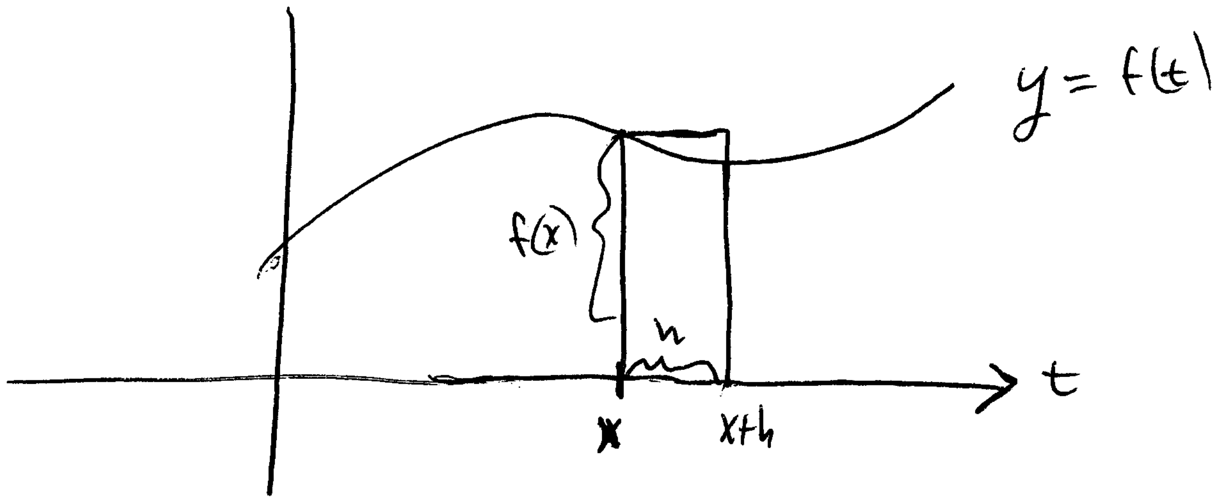
$$= x^2 \lim_{n\to\infty} \frac{n^2+n}{2n^2}$$

$$= x^2 \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2}$$

$$= \frac{1}{2} x^2$$

Note: $g'(x) = x = f(x)$.

In general,



Note: $g(x+h) - g(x) \approx$ Area of the above rectangle

$$= f(x)\,h$$

$$\Rightarrow \quad \frac{g(x+h)-g(x)}{h} \approx f(x)$$

So, perhaps,

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f(x)$$

Fundamental Theorem (Part I):

If f is cont. on $[a,b]$ then the function

$$g(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$

is cont. on $[a,b]$ and differentiable on (a,b) and

$$g'(x) = f(x)$$

Alt:

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

Eg: Let $g(x) = \int_0^x \sqrt{1+t^2}\,dt$

Then

$$g'(x) =$$

eg.: Compute

$$\frac{d}{dx}\left(\int_1^{x^4} \sec(t)\, dt \right)$$

(Hint: Don't forget the chain rule!)

- Fundamental Theorem (Part 2)

If f is cont. on $[a,b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$ on $[a,b]$, that is $F'(x) = f(x)$.

- Sketch of Proof. (see your book p. 345)

Let $g(x) = \int_a^x f(t)\,dt$. Then

$$F(x) = g(x) + C$$

Thus

$$F(b) - F(a) = (g(b)+C) - (g(a)+C) = g(b) - g(a)$$

$$= g(b) = \int_a^b f(t)\,dt \quad \blacksquare$$

e.g. Evaluate

$$\int_{-2}^{1} x^2 \, dx$$

Notation:

$$F(x)\Big]_a^b = F(b) - F(a)$$

e.g.

$$x^2 \Big|_1^3 = 3^2 - 1^2 = 9 - 1 = 8.$$

e.g.: Find the area under the cosine curve from 0 to b where $0 \le b \le \frac{\pi}{2}$.

BAD Use of Fund Thm!

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \left.\frac{-1}{x}\right]_{-1}^{1} = \frac{-1}{1} - \frac{-1}{-1}$$

$$= -1 - 1 = -2$$

What is wrong with this example?

<u>Fund. Thm:</u>

Suppose f is cont. on $[a, b]$.

1.) If $g(x) = \int_a^x f(t)\,dt$, then
$$g'(x) = f(x)$$

2.) If $F'(x) = f(x)$ then
$$\int_a^b f(x)\,dx = F(b) - F(a).$$

$$\left(\begin{array}{l} \text{If } F'(x) = f(x) \text{ then} \\ \int_a^x f(t)\,dt = F(x) - \underbrace{F(a)}_{\text{constant.}} \end{array} \right)$$