

MTHSC 206 SECTION 13.3 –DOT PRODUCTS AND PROJECTIONS

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DEFINITION

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$$(1, 2, -1) \cdot (2, 3, -2) = (1)(2) + (2)(3) + (-1)(-2) = 2 + 6 + 2 = 10$$

PROPERTIES OF THE DOT PRODUCT

Suppose that $a, b, c \in \mathbb{R}^3$ and $d \in \mathbb{R}$.

- ① $a \cdot a = |a|^2$.
- ② $a \cdot b = b \cdot a$.
- ③ $a \cdot (b + c) = a \cdot b + a \cdot c$.
- ④ $(da) \cdot b = d(a \cdot b) = a \cdot (db)$.
- ⑤ $0 \cdot a = 0$.

THE GEOMETRY OF DOT PRODUCTS

THEOREM

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where θ is the angle between a and b .

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The angle between $(1, 1, 1)$ and $(1, 1, 0)$ is $\theta = \arccos(\frac{2}{\sqrt{6}})$.

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DIRECTION ANGLES

Suppose that $a \in \mathbb{R}^3$ and that α, β and γ are the angles between a and the x -, y - and z -axes (respectively). Then,

$$\cos(\alpha) = \frac{a_1}{|a|}, \quad \cos(\beta) = \frac{a_2}{|a|} \quad \text{and} \quad \cos(\gamma) = \frac{a_3}{|a|}.$$

NOTE

Suppose that $a \in \mathbb{R}^3$ is a vector and that α, β and γ are its position angles. Then,

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- ③ $\frac{1}{|a|}a = (\cos(\alpha), \cos(\beta), \cos(\gamma)).$

DEFINITION

Suppose that $a = \vec{PQ}$ and $b = \vec{PR}$ ($Q \neq R$). Let $\mathcal{L}$ be a line which is perpendicular to the line containing a and passes through R . Let S be the intersection point of $\mathcal{L}$ with the line containing a . Then the vector $\vec{PS}$ is the vector projection of b onto a and is denoted by $\text{Proj}_a(b)$.

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DEFINITION

The scalar projection of b onto a (or component of b along a) is defined to be the signed magnitude of $\text{Proj}_a(b)$ and is denoted by $\text{comp}_a(b)$.

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Suppose that a and b are vectors and that θ is the angle between them.

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EXAMPLE

Let $a = (1, 1, 0)$ and $b = (1, 1, 1)$. Compute the vector and scalar projections of b onto a .

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So, if θ is the angle between the force vector F and the displacement vector D , then

$$W = (|F| \cos(\theta))|D| = |F||D| \cos(\theta) = F \cdot D.$$