

MTHSC 206 SECTION 13.4 –CROSS PRODUCTS

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DEFINITION

Suppose that $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_2)$ are vectors. We define the cross products $a \times b$ of a and b as

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
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NOTE

This definition will only work for 3 dimensional vectors.

EXERCISE

- ① Let $a = (1, 3, 2)$ and $b = (2, -1, 1)$. Compute $a \times b$.

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PROOF.

$$(a \times b) \cdot a$$

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$$\begin{aligned}(a \times b) \cdot a \\&= ((a_2b_3 - a_3b_2), (a_3b_1 - a_1b_3), (a_1b_2 - a_2b_1)) \cdot (a_1, a_2, a_3)\end{aligned}$$

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You can check orthogonality to b .



THEOREM

Suppose that $a, b \in \mathbb{R}^3$ and that θ is the angle between them. Then $|a \times b| = |a||b|\sin(\theta)$.

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COROLLARY

Two nonzero vectors $a, b \in \mathbb{R}^3$ are parallel if and only if $a \times b = 0$.

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The area of the parallelogram determined by $a, b \in \mathbb{R}^3$ is given by $|a \times b|$.

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EXERCISE

Find the area of the triangle with vertices $P = (1, 1, 1)$, $Q = (1, 2, 1)$ and $R = (2, 2, 3)$.

NOTE

$$\begin{array}{lll} i \times j = k & j \times k = i, & k \times i = j. \\ j \times i = -k & k \times j = -i, & i \times k = -j. \end{array}$$

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PROPERTIES OF $\times$

Suppose that $a, b, c \in \mathbb{R}^3$ and $d \in \mathbb{R}$. Then,

- ① $a \times b = -b \times a$.
- ② $(da) \times b = d(a \times b) = a \times (db)$.
- ③ $a \times (b + c) = a \times b + a \times c$.
- ④ $(a + b) \times c = a \times c + b \times c$.
- ⑤ $a \cdot (b \times c) = (a \times b) \cdot c$.
- ⑥ $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$. In particular $\times$ is not associative.

VOLUMES AND TRIPLE PRODUCTS

THEOREM

The volume of the parallelepiped determined by $a, b, c \in \mathbb{R}^3$ is given by $V = |a \cdot (b \times c)|$.

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$$V = (\text{Area of parallelogram determined by } b \text{ and } c) * (\text{height})$$

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The volume of the parallelepiped determined by $a, b, c \in \mathbb{R}^3$ is given by $V = |a \cdot (b \times c)|$.

PROOF.

$$\begin{aligned} V &= (\text{Area of parallelogram determined by } b \text{ and } c) * (\text{height}) \\ &= (\text{height}) * |b \times c| \\ &= |\text{Proj}_{b \times c}(a)| * |b \times c| \end{aligned}$$

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EXERCISE

Use the vector triple product to show that the vectors $a = (1, 1, 1)$, $b = (1, 2, 1)$ and $c = (2, 1, 2)$ are coplanar.

APPLICATION: TORQUE

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Note that

$$|\tau| = |r \times F| = |r||F| \sin(\theta),$$

where θ is the angle between F and r .