

MTHSC 206 SECTION 16.8 – TRIPLE INTEGRALS IN SPHERICAL COORDINATES

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DEFINITION

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NOTE

We note that $\rho \geq 0$, $0 \leq \theta < 2\pi$ and $0 \leq \phi \leq \pi$.

FACT

The relationship between the spherical coordinates (ρ, θ, ϕ) and the Euclidean coordinates (x, y, z) is given by

$$x = \rho \sin(\phi) \cos(\theta), \quad y = \rho \sin(\phi) \sin(\theta), \quad z = \rho \cos(\phi)$$
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EXAMPLE

- 1 The Euclidean coordinates of the point with spherical coordinates $(3, \frac{\pi}{3}, \frac{\pi}{4})$ are $(\frac{3\sqrt{6}}{4}, \frac{3\sqrt{2}}{4}, \frac{3\sqrt{2}}{2})$.

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- 2 The spherical coordinates of the point with Euclidean coordinates $(6, 6, 2\sqrt{6})$ are $(4\sqrt{6}, \frac{\pi}{4}, \frac{2\pi}{3})$.

NOTE

The spherical wedge determined by a change of angles $\Delta\theta$ from θ and $\Delta\phi$ from ϕ and a change of radius $\Delta\rho$ from ρ is $\rho^2 \sin(\phi) \Delta\rho \Delta\theta \Delta\phi$.

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Thus if E is the spherical wedge

$\{(\rho, \theta, \phi) \mid a \leq \rho \leq b; \alpha \leq \theta \leq \beta; c \leq \phi \leq d\}$, then we can derive the following formula for triple integration in spherical coordinates.

$$\iiint_E f(x, y, z) \, dV =$$

$$\int_c^d \int_\alpha^\beta \int_a^b \rho^2 \sin(\phi) f(\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi)) \, d\rho \, d\theta \, d\phi$$

EXAMPLE

Evaluate $\int \int \int_B e^{(x^2+y^2+z^2)^{3/2}} dV$ where B is the unit ball about the origin.

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Find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.