§ 1.1 Systems of linear equations.

Def: A linear equation in the variables $x_1, x_2, x_3, \ldots, x_n$ is an equation of the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

where $a_1, a_2, \ldots, a_n, b$ are real or complex #'s

- **eg.:** $5x_1 + 3x_2 + 4x_3 = 7$ is linear

$5x_1^2 + 3x_2 + 4x_3 = 8$ is not linear.

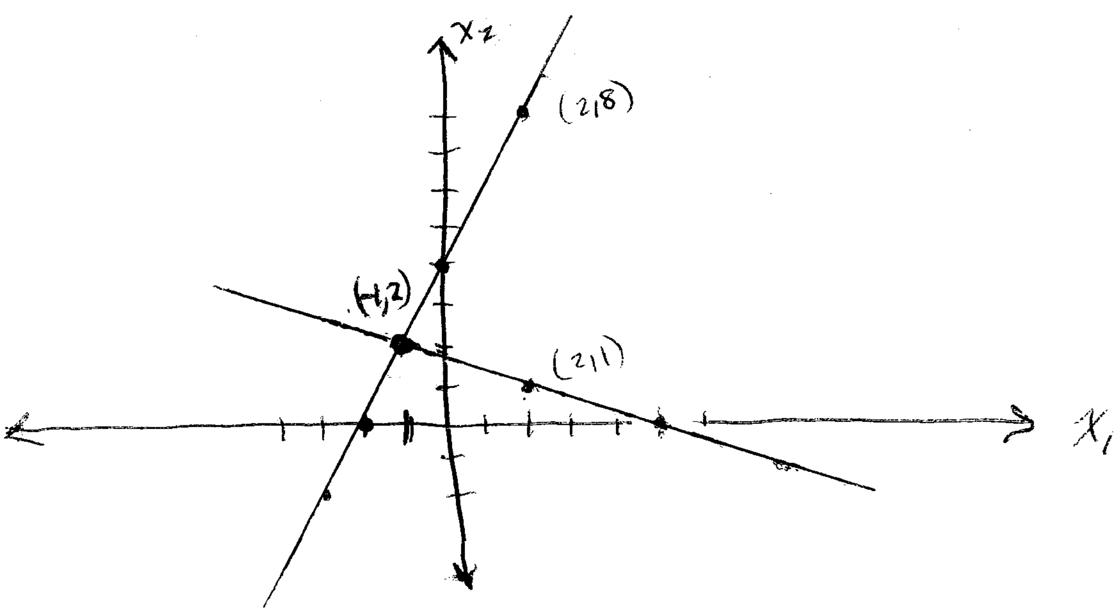
Def: A <u>system of linear equations</u> is a collection of linear equations in the same variables

Example: A system of two linear equations:

$$x_1 + 3x_2 = 5$$
$$2x_1 - x_2 = -4$$

Note: Solutions to the above system can be thought of as the intersection points of the two lines:

$$x_2 = -\tfrac{1}{3}x_1 + 5/3$$
$$x_2 = 2x_1 + 4$$

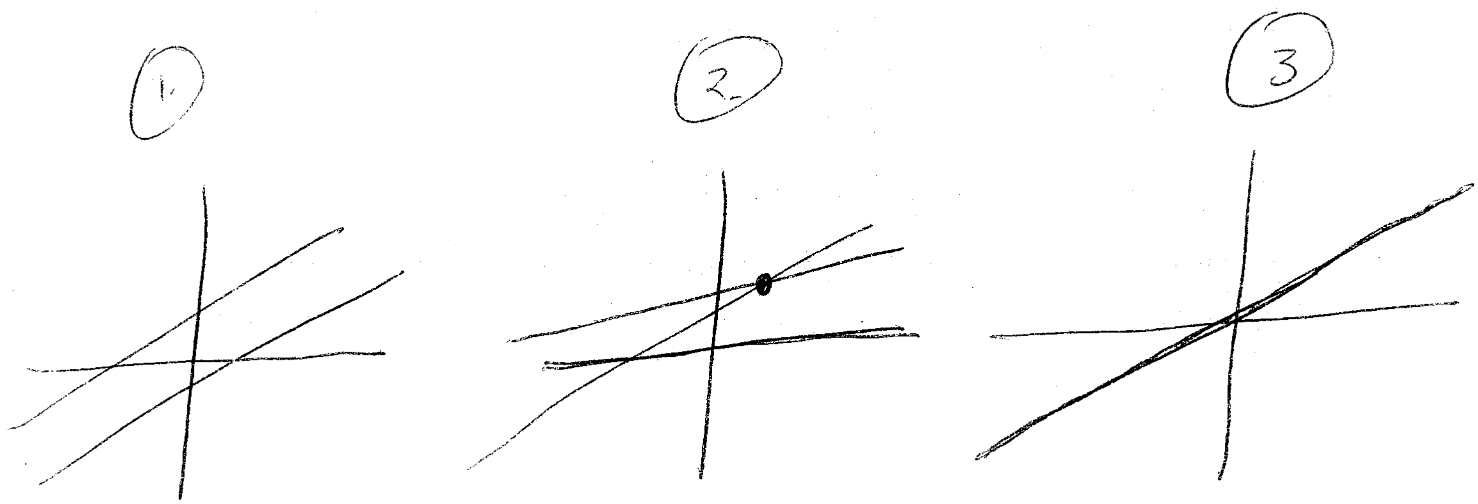


3 Possibilities for solution sets:

1.) No solutions

2.) Exactly one solution

3.) Infinitely many solutions.



Matrix Notation:

$$\begin{cases} x_1 + 3x_2 = 5 \\ 2x_1 - x_2 = -4 \end{cases}$$

has the coefficient matrix:

$$\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

and the augmented matrix

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & -4 \end{bmatrix}$$

Definition: We say a matrix is $m \times n$ provided that it has m rows and n columns.

Example:

$$\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 2 \end{bmatrix} \quad \text{is} \quad 2 \times 3$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 6 & 11 \end{bmatrix} \quad \text{is} \quad 3 \times 2.$$

Solving a system:

Example: Solve.

$$\begin{cases} x_1 + 3x_2 = 5 \\ 2x_1 - x_2 = -4 \end{cases} \qquad \begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & -4 \end{bmatrix}$$

• Note: The techniques used in the previous example correspond to the following operations on the augmented matrix. (4)

• **Elementary Row Operations:**

1.) (Replacement): Replace a row by the sum of itself and a multiple of another.

2.) (Interchange): Interchange two rows.

3.) (Scaling) Multiply a row by a nonzero constant.

Example: Use elementary row operations to solve

$$\begin{cases} x_1 + 3x_2 = 5 \\ 2x_1 - x_2 = -4 \end{cases} \qquad \begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & -4 \end{bmatrix}$$

Definition ① Two linear systems are <u>equivalent</u> provided that they have the same solution set.

② Two matrices are <u>row equivalent</u> provided that one matrix can be transformed to the other by elementary row operations.

Fact: Two systems of linear equations are equivalent if and only if their augmented matrices are row equivalent.

Example: Solve

$$\begin{cases} x_1 \qquad\quad - 3x_3 = 8 \\ 2x_1 + 2x_2 + 9x_3 = 7 \\ \qquad\quad x_2 + 5x_3 = -2 \end{cases}$$

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix} \longrightarrow$$

Existence & Uniqueness.

Fundamental Questions.

1) Is the system consistent?
   That is, does a solution exist?

2) If a solution exists, is it unique (i.e. the only one)?

Example: We have seen that

$$\begin{cases} x_1 + 3x_2 = 5 \\ 2x_1 - x_2 = -4 \end{cases}$$

has exactly one solution.

That is, a solution exists and it is unique.

Example: solve

$$\begin{cases} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases} \qquad \begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

$$\rightarrow \cdots \longrightarrow \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

Thus the original system is equivalent to

$$\begin{cases} 2x_1 - 3x_2 + 2x_3 = 1 \\ x_2 - 4x_3 = 8 \\ 0x_1 + 0x_2 + 0x_3 = 5/2 \end{cases}$$

which clearly has no solutions.

Example:

Solve
$$\begin{cases} x_1 + x_2 = 5 \\ 2x_1 + 2x_2 = 10 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 5 \\ 2 & 2 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

which is equivalent to

$$\begin{cases} x_1 + x_2 = 5 \\ 0 = 0 \end{cases}$$

This has infinitely many solutions:

Let x_1 be anything you like and take $x_2 = 5 - x_1$.

So, we see that solutions exist but are not unique.