§1.2 Row Reduction & Echelon Forms:

Definition: A matrix is in (row) echelon form provided

1) All non zero rows are above any zero rows.
2) Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3) All entries in a column below a leading entry are zeros

Additionally, a matrix is in reduced (row) echelon form provided that

4) The leading entry in each nonzero row is 1.
5) Each leading 1 is the only nonzero entry in its column.

Example:

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$ is in echelon form

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ is in reduced echelon form.

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$ is not in echelon form.

Some Examples of Matrices

a) $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 2 & 1 & 5 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 1 & -4 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 & 3 & -9 & 2 & 5 \\ 0 & 1 & 4 & 6 & 1 & 0 \end{bmatrix}$

f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- **Theorem 1:** Each matrix is row equivalent to a unique reduced echelon matrix.

- **Def:** A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

Row Reduction Algorithm:

1.) Begin w/ left most non zero column. This is a pivot column, the pivot position is at the top.

2.) Select a non zero entry in the column as the pivot. Interchange rows if necessary to move the entry into pivot position.

3.) Use row replacement operations to create zeros in all positions below the pivot.

4.) Move to the next row and apply steps 1-3 to the remaining submatrix. (i.e. the rows below and including the current one.)

5.) Scale pivots to be 1 and use row replacement to eliminate nonzero entries above the pivots.

Example: Row reduce

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix} \longrightarrow \Bigg[$$

- Solutions of Linear Equations:

Recall that there are 3 possibilities:

1.) No solution,

-e.g. $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

2.) Unique solution,

-e.g. $\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6 \end{bmatrix}$ Solution: $\begin{cases} x_1 = \\ x_2 = \\ x_3 = \end{cases}$

3. Infinitely Many Solutions.

e.g.: $\begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \equiv \begin{cases} x_1 + 2x_2 = 6 \\ x_3 = 3 \end{cases}$

We write the solution set as

$$\begin{cases} x_1 = 6 - 2x_2 \\ x_2 \text{ is free} \\ x_3 = 3 \end{cases}$$

Note:
1. This called a general solution.
2. x_2 is called a parameter or free variable.

Example: Find the general solution to the linear system with augmented matrix:

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

Existence and Uniqueness

In the previous example, it's easy to see that a solution exists. But, since we can choose any values for the free variables, the solution is <u>not</u> unique.

<u>Theorem 2</u> (pg 24) A linear system is consistent iff an echelon form of the augmented matrix has <u>no</u> row of the form $[0 \cdots 0 \; b]$ with b nonzero.

If a linear system is consistent, then the solution set contains either (i) a unique solution (no free variables) or (ii) infinitely many solutions (at least one free var.)

Solution procedure (p 24)

1. write the augmented matrix
2. row reduction algorithm $\rightarrow$ echelon form
   is the system consistent (no - stop, yes - #3)
3. echelon form $\Rightarrow$ reduced echelon form
4. write system of eqns corresp. to the matrix in 3.
5. rewrite each nonzero eqn so that its one basic variable is expressed in terms of any free variables appearing in the eqn.