§1.3 Vector Equations

- Vector: A Matrix with one column.

- Example:

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -5 \\ 9 \end{bmatrix}$$

- Note: Two vectors are equal when they have the same size & their corresponding entries are equal.

Def: We define the sum and scalar product by

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \end{bmatrix}$$

$$\alpha \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} \end{bmatrix}$$

- e.g.:

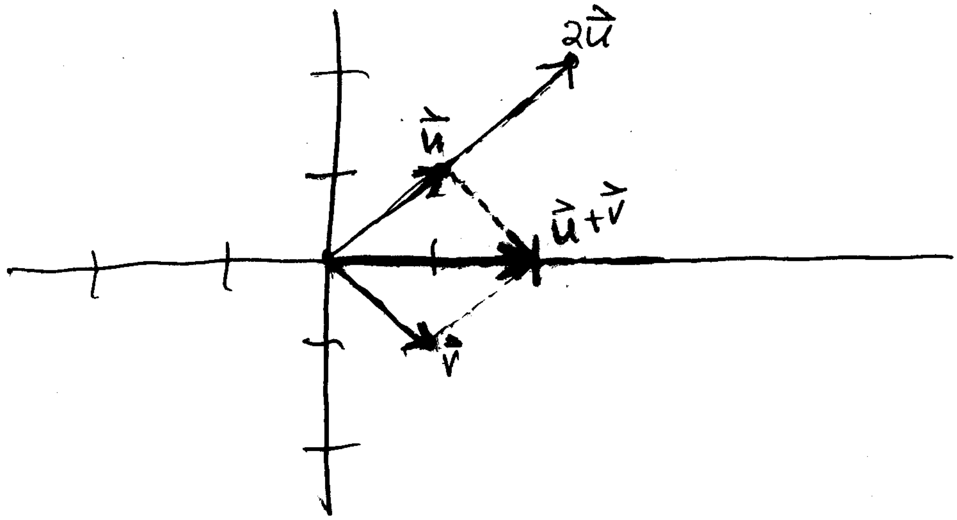
$$\begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} =$$

$$3 \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} =$$

- **Geometry:**

Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

We think geometricly of $\vec{u}, \vec{v}, 2\vec{u}, \vec{u}+\vec{v}$ as



- **Parallelogram Rule for addition.**

If $\vec{u}, \vec{v} \in \mathbb{R}^2$, Then $\vec{u}+\vec{v}$ corresponds to the 4^{th} vertex of the parallelogram whose other vertices are $\vec{0}$, $\vec{u}$ & $\vec{v}$.

- **Example:** Let $\vec{u} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$,

Display $\vec{u}, -2/3\,\vec{u}, \vec{v}, -2/3\,\vec{u}+\vec{v}$ on a graph.

- In general we will consider vectors

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \text{ in } \mathbb{R}^n,$$

We have $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$,

- <u>Properties of $\mathbb{R}^n$.</u>

1.) $u+v = v+u$
2.) $(u+v)+w = u+(v+w)$
3.) $u+0 = 0+u = u$
4.) $u+(-u) = \vec{0} = -u+u$
$(-\vec{u} := (-1)\vec{u})$
5.) $c(u+v) = cu+cv$
6.) $(c+d)u = cu+du$
7.) $c(du) = (cd)u$
8.) $1 \cdot u = u$

- <u>Linear Combinations.</u>

Given $v_1, \ldots, v_p \in \mathbb{R}^n$ and $c_1, \ldots, c_n \in \mathbb{R}$, the vector

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

is called the linear combination of the vectors $v_1, \ldots, v_p$ with weights $c_1, \ldots, c_n$.

<u>example:</u>

$$2\begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 3 \\ -8 \end{bmatrix} =$$

is a linear combination of $\begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -8 \end{bmatrix}$ w/ weights $2, 3, -2$.

Geometry: Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Show all linear combinations of $\vec{u}$ & $\vec{v}$ on a graph.

Examples. Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ & $b = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$

Is b a lin. comb. of v_1 & v_2?

- ## Vector Equation,

A vector equation

$$x_1 \vec{v}_1 + \cdots + x_n \vec{v}_n = \vec{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} | & | & & | & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n & \vec{b} \\ | & | & & | & | \end{bmatrix}$$

In particular $\vec{b}$ is a lin. comb. of $\vec{v}_1, \dots, \vec{v}_n$ if and only if the system is consistent.

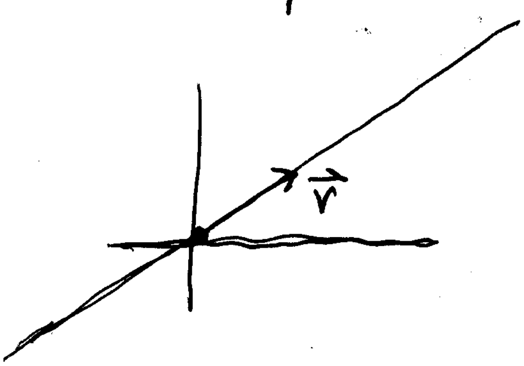
- **Def:** S $\vec{v}_1, \dots, \vec{v}_p \in \mathbb{R}^n$. We define

$$\text{Span}(\vec{v}_1, \dots, \vec{v}_p) = \{ c_1\vec{v}_1 + \cdots + c_p\vec{v}_p : c_1, c_2, \dots, c_p \in \mathbb{R} \}.$$

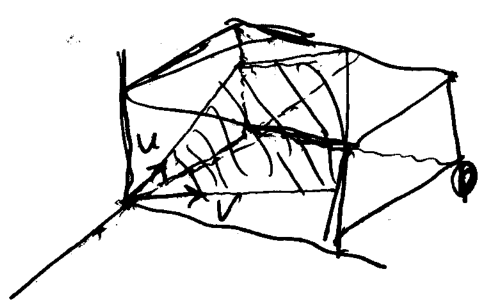
That is $\text{Span}(\vec{v}_1, \dots, \vec{v}_p)$ is the set of all linear combinations of $\vec{v}_1, \dots, \vec{v}_p$.

- ## Geometry:

$\text{span}(\vec{v})$



$\text{Span}(\vec{u}, \vec{v})$



- Example: Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} -9 \\ -30 \\ 31 \end{bmatrix}$

Span$(\vec{v}_1, \vec{v}_2)$ is a plane in $\mathbb{R}^3$.
Is $\vec{b}$ is that plane?

- see application on p. 36.