§ 1.4 The Matrix Eq. $A\vec{x} = \vec{b}$

Def:
If A is an $m \times n$ matrix with columns $\vec{a}_1, \ldots, \vec{a}_n$ and $\vec{x} \in \mathbb{R}^n$, then the product of A and $\vec{x}$ denoted $A\vec{x}$ is defined to be

$$A\vec{x} = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n$$

- Note: $A\vec{x} \in \mathbb{R}^m$.

- eg.:

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} =$$

Thm 3: If A is an $m \times n$ matrix with columns $\vec{a}_1, \ldots, \vec{a}_n$ and if $\vec{b} \in \mathbb{R}^m$ the matrix eq.

$$A\vec{x} = \vec{b}$$

has the same solutions as the vector eq.

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n = \vec{b}.$$

which in turn has the same solutions as the linear system w/ augmented matrix

$$\begin{bmatrix} | & | & & | & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n & \vec{b} \\ | & | & & | & | \end{bmatrix}.$$

Pf: This follows from our definition of mult of matrices and vectors ■

Thm: The eq. $A\vec{x} = \vec{b}$ has a solution iff $\vec{b}$ is a linear combination of the columns of A.

Example: For what vectors $\vec{b}$ is the eq. $A\vec{x} = \vec{b}$ solvable when

$$A = \begin{bmatrix} 1 & 8 & -2 \\ 7 & 2 & 3 \\ -2 & 13 & -13 \end{bmatrix} \quad ?$$

• Thm 4: Let A be an $m \times n$ matrix. The following statements are equivalent.

1.) For each $\vec{b} \in \mathbb{R}^m$, the eq. $A\vec{x} = \vec{b}$ has a solution.
2.) Each $\vec{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.
3.) The columns of A span $\mathbb{R}^m$.
4.) A has a pivot position in every row.

(Note: Part 4 refers to the coeff. matrix A not the augmented matrix $[A \; \vec{b}]$.)

Pf: Statements 1. 2. & 3. are equivalent by definition.

We will now show that 1. and 4. are equivalent.

Suppose that U is the reduced echelon form of A.

Then

$$[A \;\; \vec{b}] \sim \cdots \sim [U \;\; \vec{d}]$$

for some $\vec{d}$.

- Computing $A\vec{x}$:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{bmatrix}$$

- example:

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 1 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 11 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Thm. 5.:

If A is $m \times n$ and $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then

1.) $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$

2.) $A(c\vec{u}) = c(A\vec{u})$

Pf:

§1.5 Solution Sets of Linear Systems

- Def: A system of linear eq.'s is called homogeneous if it can be written as
$$A\vec{x} = \vec{0}.$$

- Note: $\vec{x} = \vec{0}$ is a solution to $A\vec{x} = \vec{0}$. It is called the trivial solution.

- Fact: The homogenous eq. $A\vec{x} = \vec{0}$ has a non trivial solution iff it has free variables.

- Pf:

- Example: Determine if the following homogeneous system has nontrivial solutions.
$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 0 \\ 5x_2 - x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \end{aligned}$$

Example (cont.) Describe the solution set.

- *Note*: The solution set of $A\vec{x} = \vec{0}$ can always be written as $\mathrm{Span}(\vec{v}_1, \dots, \vec{v}_p)$ for some vectors $\vec{v}_1, \dots, \vec{v}_p$.

- *Def*: An equation of the form

$$\vec{x} = s_1 \vec{v}_1 + s_2 \vec{v}_2 + \dots + s_k \vec{v}_k$$

is said to be in vector parametric form.

- *Example*: Describe all solutions to $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

Theorem 6: $\not S$ $A\vec{x}=\vec{b}$ has a solution, $\vec{p}$.

Then all solutions have the form

$$\vec{w} = \vec{p} + \vec{v}_h$$

where $\vec{v}_h$ is a solution to $A\vec{x}=\vec{0}$.

(- ie, If $\vec{p}$ is any solution to $A\vec{x}=\vec{b}$ and the solution set of $A\vec{x}=\vec{0}$ is $\text{Span}(\vec{v}_1,\dots,\vec{v}_k)$ then the solution set of $A\vec{x}=\vec{b}$ is

$$\vec{p} + \text{Span}(\vec{v}_1,\dots,\vec{v}_k)\ )$$

- Pf:

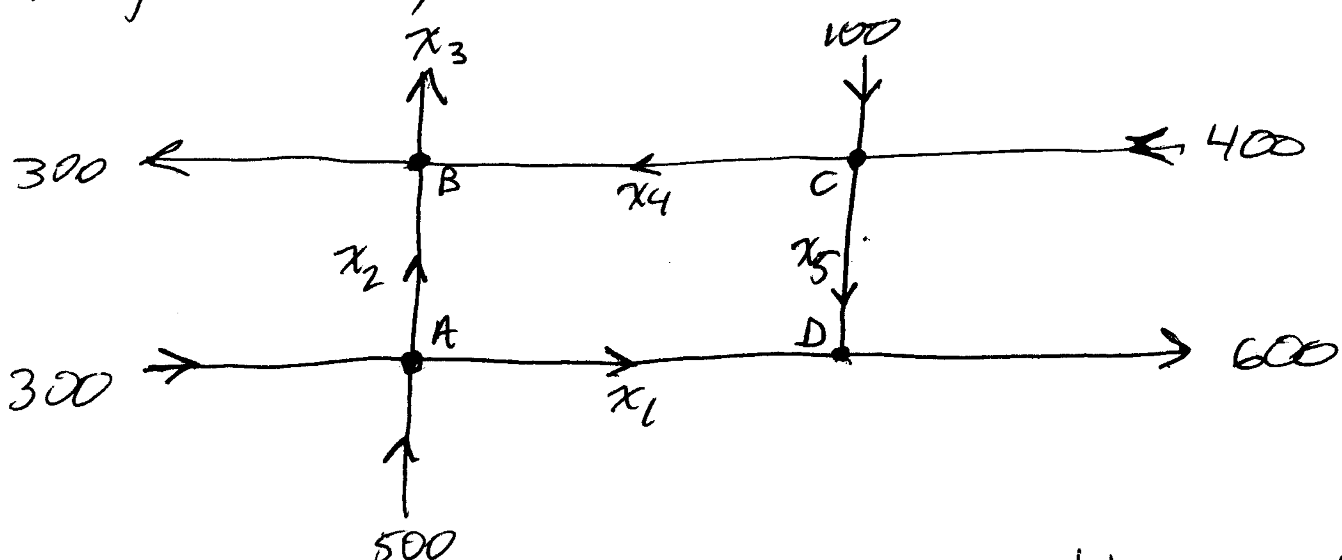
§1.6 Applications.

* Read this section on your own.

Example: Balance the eq.

$$KMnO_4 + MnSO_4 + H_2O \rightarrow MnO_2 + K_2SO_4 + H_2SO_4$$

• Example: Network flow.
The following depicts traffic flow along one-way streets in Baltimore.



Determine the general flow pattern of the network.

§1.7 Linear Independence.

- **Def**: An indexed set of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ is <u>linearly independent</u> if the vector eq.

$$x_1 \vec{v}_1 + \dots + x_p \vec{v}_p = \vec{0}$$

has only the trivial solution $(\vec{x} = \vec{0})$.

Otherwise if $\exists\ c_1, \dots, c_p \in \mathbb{R}$, not all zero, s.t.

$$c_1 \vec{v}_1 + \dots + c_p \vec{v}_p = \vec{0}$$

then the set is <u>linearly dependent</u>.

Example:

are $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -1 \\ 7 \\ 21 \end{bmatrix}$ LI?

If not, find a dependence.

Note! The columns of a matrix A are L.I. iff $A\vec{x} = \vec{0}$ has only the trivial solution.

Example:

are the columns of $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 2 \\ 5 & 3 & 3 \end{bmatrix}$ LI?

Note: (1) A set w/ only one vector $\vec{v}$ is L.I. iff $\vec{v} \neq \vec{0}$.

(2) A set of two vectors is L.I. iff neither vector is a multiple of the other.

Pf:

Thm 7: $S = \{v_1, \dots, v_p\}$; $p \geq 2$ is L.D. iff one of the vectors in S is a lin. comb. of the others.
In fact S is L.D. iff either $v_1 = \vec{0}$ or $\exists\, j$ s.t. $\vec{v}_j$ is a lin. comb. of $\vec{v}_1, \dots, \vec{v}_{j-1}$.

Pf:

Example:

Let $\vec{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$

Describe span$(\vec{u}, \vec{v})$.

For this particular $\vec{u}, \vec{v}$ we have
$\vec{w} \in$ span$(\vec{u}, \vec{v})$ iff $\{u, v, w\}$ is L.D. Explain.

Thm 8: If a set contains more vectors than there are entries in the vectors then it is L.D.

Pf:

- Thm 9: If $\vec{0} \in S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ then S is LD.

- Pf: