MTHSC 3110 Section 1.4 – The Matrix Equation $A\vec{x} = \vec{b}$

Kevin James

Definition

If A is an $m \times n$ matrix with columns $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$, and $\vec{x} \in \mathbb{R}^n$, then the product of A and $\vec{x}$, which we denote by $A\vec{x}$, is defined to be

$$A\vec{x} = \begin{pmatrix} | & | & \ldots & | \\ \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \\ | & | & \ldots & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

Definition

If A is an $m \times n$ matrix with columns $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$, and $\vec{x} \in \mathbb{R}^n$, then the product of A and $\vec{x}$, which we denote by $A\vec{x}$, is defined to be

$$A\vec{x} = \begin{pmatrix} | & | & \cdots & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & \cdots & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

Note

1. $A\vec{x} \in \mathbb{R}^m$.
2. $A\vec{x}$ is the linear combination of the columns of A with weights $x_1, x_2, \ldots, x_n$.

Example

$$\begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} =$$

Theorem

If A is an $m \times n$ matrix with columns $\vec{a}_1, \vec{a}_2, \ldots \vec{a}_n$, and if $\vec{b} \in \mathbb{R}^m$, then the matrix equation $A\vec{x} = \vec{b}$ has the same set of solutions as the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$

which in turn has the same set of solutions as the linear system with augmented matrix

$$\begin{pmatrix} | & | & \ldots & | & | \\ \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n & \vec{b} \\ | & | & \ldots & | & | \end{pmatrix}$$

Theorem

If A is an $m \times n$ matrix with columns $\vec{a}_1, \vec{a}_2, \dots \vec{a}_n$, and if $\vec{b} \in \mathbb{R}^m$, then the matrix equation $A\vec{x} = \vec{b}$ has the same set of solutions as the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$

which in turn has the same set of solutions as the linear system with augmented matrix

$$\begin{pmatrix} | & | & \dots & | & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n & \vec{b} \\ | & | & \dots & | & | \end{pmatrix}$$

Proof.

This follows from the definition of multiplication of a matrix and a vector. $\square$

Theorem

The equation $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is a linear combination of the columns of A.

THEOREM

The equation $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is a linear combination of the columns of A.

PROOF.

This follows from the definition of multiplication of a matrix and a vector. $\square$

Example

For which vectors $\vec{b}$ is the equation $A\vec{x} = \vec{b}$ solvable, where

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 7 & 2 & 3 \\ -2 & 13 & -13 \end{pmatrix}?$$

Theorem

Let A be an $m \times n$ matrix. The following statements are equivalent:

1. *For each $\vec{b} \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.*
2. *Each $\vec{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.*
3. *The columns of A span $\mathbb{R}^m$.*
4. *In the row reduction process, A has a pivot position in every row*

Theorem

Let A be an $m \times n$ matrix. The following statements are equivalent:

1. *For each $\vec{b} \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.*
2. *Each $\vec{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.*
3. *The columns of A span $\mathbb{R}^m$.*
4. *In the row reduction process, A has a pivot position in every row*

Note

Part 4 refers to the coefficient matrix A, not the augmented matrix $[A \quad \vec{b}]$.

Proof.

Statements 1, 2 and 3 are equivalent by definition.

Proof.

Statements 1, 2 and 3 are equivalent by definition.
So if we show that 1 and 4 are equivalent, we will be done.

Proof.

Statements 1, 2 and 3 are equivalent by definition.
So if we show that 1 and 4 are equivalent, we will be done.

Suppose that U is the reduced echelon form of A.

Proof.

Statements 1, 2 and 3 are equivalent by definition.
So if we show that 1 and 4 are equivalent, we will be done.

Suppose that U is the reduced echelon form of A.
Then for some vector $\vec{d}$, we have

$$[A \quad \vec{b}] \sim \cdots \sim [U \quad \vec{d}]$$

Proof.

Statements 1, 2 and 3 are equivalent by definition.
So if we show that 1 and 4 are equivalent, we will be done.

Suppose that U is the reduced echelon form of A.
Then for some vector $\vec{d}$, we have

$$[A \quad \vec{b}] \sim \cdots \sim [U \quad \vec{d}]$$

If statement 4 is true, then since U has a pivot in every row, we clearly don't have a row of zeros in U with a non-zero element in $\vec{d}$.

PROOF.

Statements 1, 2 and 3 are equivalent by definition.
So if we show that 1 and 4 are equivalent, we will be done.

Suppose that U is the reduced echelon form of A.
Then for some vector $\vec{d}$, we have

$$[A \quad \vec{b}] \sim \cdots \sim [U \quad \vec{d}]$$

If statement 4 is true, then since U has a pivot in every row, we clearly don't have a row of zeros in U with a non-zero element in $\vec{d}$. Hence the equation is solvable no matter what $\vec{b}$ is.

Proof.

Statements 1, 2 and 3 are equivalent by definition.
So if we show that 1 and 4 are equivalent, we will be done.

Suppose that U is the reduced echelon form of A.
Then for some vector $\vec{d}$, we have

$$[A \quad \vec{b}] \sim \cdots \sim [U \quad \vec{d}]$$

If statement 4 is true, then since U has a pivot in every row, we clearly don't have a row of zeros in U with a non-zero element in $\vec{d}$.
Hence the equation is solvable no matter what $\vec{b}$ is.
Thus, if statement 4 is true, then 1 is true.

Proof.

Statements 1, 2 and 3 are equivalent by definition.
So if we show that 1 and 4 are equivalent, we will be done.

Suppose that U is the reduced echelon form of A.
Then for some vector $\vec{d}$, we have

$$[A \quad \vec{b}] \sim \cdots \sim [U \quad \vec{d}]$$

If statement 4 is true, then since U has a pivot in every row, we clearly don't have a row of zeros in U with a non-zero element in $\vec{d}$.
Hence the equation is solvable no matter what $\vec{b}$ is.
Thus, if statement 4 is true, then 1 is true.

Conversely, if 4 is false, U has a zero row.

Proof.

Statements 1, 2 and 3 are equivalent by definition.
So if we show that 1 and 4 are equivalent, we will be done.

Suppose that U is the reduced echelon form of A.
Then for some vector $\vec{d}$, we have

$$[A \quad \vec{b}] \sim \cdots \sim [U \quad \vec{d}]$$

If statement 4 is true, then since U has a pivot in every row, we clearly don't have a row of zeros in U with a non-zero element in $\vec{d}$.
Hence the equation is solvable no matter what $\vec{b}$ is.
Thus, if statement 4 is true, then 1 is true.

Conversely, if 4 is false, U has a zero row.
Let $\vec{d}$ be the vector with a 1 in that row, and zeros elsewhere.

Proof.

Statements 1, 2 and 3 are equivalent by definition.
So if we show that 1 and 4 are equivalent, we will be done.

Suppose that U is the reduced echelon form of A.
Then for some vector $\vec{d}$, we have

$$[A \quad \vec{b}] \sim \cdots \sim [U \quad \vec{d}]$$

If statement 4 is true, then since U has a pivot in every row, we clearly don't have a row of zeros in U with a non-zero element in $\vec{d}$.
Hence the equation is solvable no matter what $\vec{b}$ is.
Thus, if statement 4 is true, then 1 is true.

Conversely, if 4 is false, U has a zero row.
Let $\vec{d}$ be the vector with a 1 in that row, and zeros elsewhere.
Reversing the row reduction process we obtain a vector $\vec{b}$ for which $A\vec{x} = \vec{b}$ has no solution.

Proof.

Statements 1, 2 and 3 are equivalent by definition.
So if we show that 1 and 4 are equivalent, we will be done.

Suppose that U is the reduced echelon form of A.
Then for some vector $\vec{d}$, we have

$$[A \quad \vec{b}] \sim \cdots \sim [U \quad \vec{d}]$$

If statement 4 is true, then since U has a pivot in every row, we clearly don't have a row of zeros in U with a non-zero element in $\vec{d}$.
Hence the equation is solvable no matter what $\vec{b}$ is.
Thus, if statement 4 is true, then 1 is true.

Conversely, if 4 is false, U has a zero row.
Let $\vec{d}$ be the vector with a 1 in that row, and zeros elsewhere.
Reversing the row reduction process we obtain a vector $\vec{b}$ for which $A\vec{x} = \vec{b}$ has no solution.
Hence if statement 4 is false, so is statement 1. $\square$

In the computation of

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{pmatrix}$$

we see that

$$b_i = \sum_{j=1}^{n} a_{ij} x_j = a_{i1}x_1 + a_{i2}x_2 + \dots a_{in}x_n.$$

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & -1 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 11 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$$

Theorem

If A is an $m \times n$ matrix and $\vec{u}, \vec{v}$ are vectors in $\mathbb{R}^n$ and $c \in \mathbb{R}$ is a scalar then

1. $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$
2. $A(c\vec{u}) = c(A\vec{u})$