§1.8 Linear Transformations.

Def: A function $T: \mathbb{R}^n \to \mathbb{R}^m$ is sometimes called a <u>transformation.</u> $\mathbb{R}^n$ is the <u>domain.</u> $\mathbb{R}^m$ is the <u>codomain.</u> The set $\{T(\vec{x}) : \vec{x} \in \mathbb{R}^n\}$ is called the <u>image</u> or <u>range</u> of T.

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 5 & -1 \end{bmatrix}$

Define $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$.

1.) $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) =$

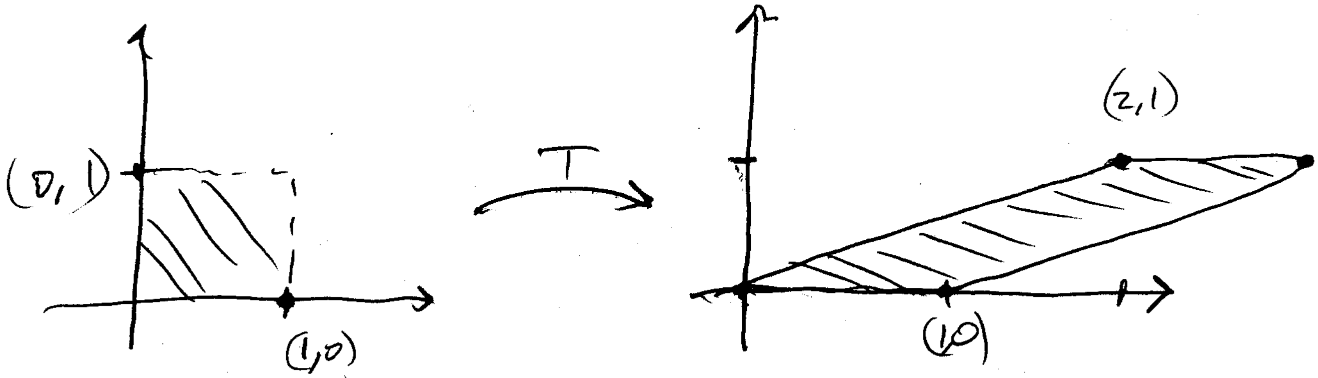
2.) Find all $\vec{x} \in \mathbb{R}^2$ s.t. $T(\vec{x}) = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$.

Example: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Then $T(\vec{x}) = A\vec{x}$ is called a projection, because

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

Example: Shear Transformation.

$$T(\vec{x}) = A\vec{x} \quad \text{where} \quad A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$



Definition:

A transformation T is linear if:

1.) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \forall \vec{u}, \vec{v} \in \text{domain}(T)$

2.) $T(c\vec{u}) = c\,T(\vec{u}) \quad \forall \vec{u} \in \text{domain}(T);\ c, \text{scalar}.$

Note: If T is linear then

1.) $T(\vec{0}) = \vec{0}$

2.) $T(c\vec{u} + d\vec{v}) = c\,T(\vec{u}) + d\,T(\vec{v})$.

3.) $T\left(\sum_{i=1}^{p} c_i \vec{v}_i\right) = \sum_{i=1}^{p} c_i T(\vec{v}_i)$.

Pf:

<u>Examples</u>: Given $0 \le r \le 1$, the map $T_r: \mathbb{R}^2 \to \mathbb{R}^2$

$$T_r(\vec{x}) = r\vec{x}$$

is called a <u>contraction</u>.

If $r > 1$ then T is a <u>dilation</u>.

Show that T_r is <u>linear</u>.

§ 1.9 The matrix of a linear transformation.

<u>Example</u>: S $T: \mathbb{R}^2 \to \mathbb{R}^3$ is linear

and $T(\vec{e}_1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $T(\vec{e}_2) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

where $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Give

a complete description of T.

<u>Theorem 10</u>:

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then $\exists !\ A$ s.t.

$$T(\vec{x}) = A\vec{x} \quad \forall\, \vec{x} \in \mathbb{R}^n.$$

In fact A is the $m \times n$ matrix whose $j^{\underline{th}}$ column is $T(\vec{e}_j)$.

$$\left(\text{i.e.} \quad A = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ | & | & & | \end{bmatrix} \right).$$

<u>Proof</u>:

<u>Example</u>: Find the matrix for $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{x}) = 3\vec{x}$.

• (See pp. 85–87 for other examples.)

- **Definition:** A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be <u>onto</u> $\mathbb{R}^m$ if each $\vec{b} \in \mathbb{R}^m$ is the image of at least one $\vec{x} \in \mathbb{R}^n$.
  (T is onto provided $\forall \vec{b} \in \mathbb{R}^m, \exists \vec{x} \in \mathbb{R}^n$ s.t. $T(\vec{x}) = \vec{b}$).

- **Definition:** A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is <u>one to one</u> provided that each $\vec{b} \in \mathbb{R}^m$ is the image of at most one $\vec{x} \in \mathbb{R}^n$
  (i.e. $T\vec{x} = T\vec{y} \Rightarrow \vec{x} = \vec{y}$)

- **Example:** Let T be the linear transformation w/ matrix

$$A = \begin{bmatrix} 1 & 2 & 5 & 6 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Is T 1-1?

Thm #: $T: \mathbb{R}^n \to \mathbb{R}^m$ is 1-1 <u>iff</u> $T\vec{x} = \vec{0}$ has only the trivial solution.

- **Pf:**

Theorem 12: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be linear and let A be its matrix. Then,

a) T is onto iff the span of the columns of A is $\mathbb{R}^m$.

b) T is 1-1 iff the columns of A are LI.

Pf:

Example: Let $T(x_1, x_2) = (3x_1 + x_2,\ 5x_1 + 7x_2,\ x_1 + 3x_2)$

Is T 1-1, onto.