*** Read the applications in 1.10.**

§ 2.1 Matrix Operations.

$$A = \begin{array}{cl} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & & & & & \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & & a_{mn} \end{bmatrix} & \begin{array}{l} \\ \\ \\ \to i^{th} \text{ row} \\ \\ \\ \end{array} \\ \uparrow & \\ j^{th} \text{ column.} & \end{array}$$

Def! If $A = [\vec{a}_1 \cdots \vec{a}_n]$ where $\vec{a}_1, \ldots, \vec{a}_n$ are column vectors and $B = [\vec{b}_1 \cdots \vec{b}_n]$, then we define

$$A + B = [(\vec{a}_1 + \vec{b}_1) \cdots (\vec{a}_n + \vec{b}_n)]$$ and,

$$cA = [c\vec{a}_1 \cdots c\vec{a}_n]$$

e.g.

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} & \\ & \\ & \end{bmatrix}$$

$$3 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

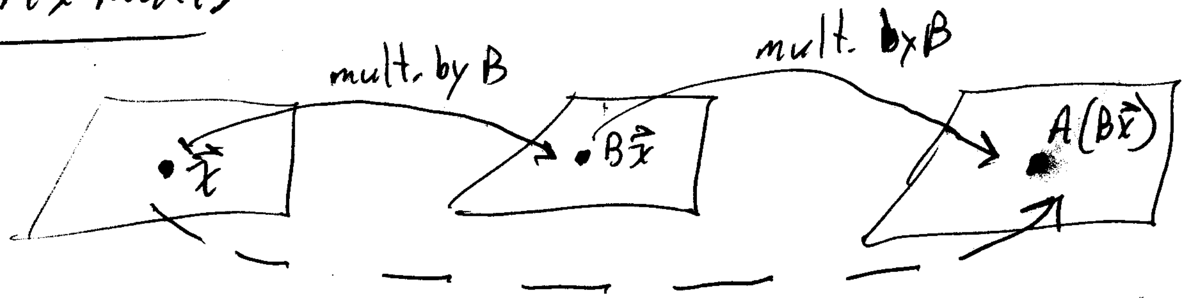
Thm 1: Suppose A, B, C are $m \times n$ and $r, s \in \mathbb{R}$.

Then

1.) $A + B = B + A$
2.) $(A+B)+C = A+(B+C)$
3.) $A + 0 = A$
4.) $r(A+B) = rA + rB$
5.) $(r+s)A = rA + sA$
6.) $r(sA) = (rs)A$.

Pf!

Matrix Mult:



We want to define a matrix AB so that mult. by AB is the same as first multiplying by B then by A.

(ie we want $(AB)\vec{x} = A(B\vec{x})$)

, Derive AB:

Def: If A is $m \times n$ and B is $n \times p$ w/ columns $\vec{b}_1 \cdots \vec{b}_p$ then we define

$$AB = \left[\qquad\qquad\qquad\qquad \right]$$

e.g.

$$\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 6 \\ 7 & 5 & 3 \end{bmatrix} = \left[\qquad\qquad\qquad \right].$$

, Note: Each column of AB is a linear combination of the columns of A using weights in the corresponding column of B.

, Row-Column Rule: If A is $m \times n$ & B is $n \times p$ then

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Note:

$$\mathrm{Row}_i(AB) = \mathrm{Row}_i(A) \cdot B$$

Why?

Thm 2:

1.) $(AB)C = A(BC)$

2.) $A(B+C) = AB + AC$

3.) $(B+C)A = BA + CA$

4.) $r(AB) = (rA)B = A(rB) \quad \forall r \in \mathbb{R}$

5.) $I_m A = A = A I_n$

I_m: m×m identity matrix

I_n: n×n identity matrix

$$\left(I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix} = [\vec{e}_1 \cdots \vec{e}_n] \right)$$

Pf:

- **Warning:** 1. In general, $AB \neq BA$.
  2. $AB = AC \not\Rightarrow B = C$
  3. $AB = 0 \not\Rightarrow A = 0$ or $B = 0$.

- **eg:**

$$\begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

- **Def:** 1) $A^k = \underbrace{A \cdot A \cdot \ldots \cdot A}_{k \text{ times.}}$

2) The transpose A^T of A is defined by

$$[A^T]_{ij} = A_{ji}$$

- **eg:**

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = [$$

Thm 3:

1.) $(A^T)^T = A$.

2.) $(A+B)^T = A^T + B^T$

3.) $(rA)^T = rA^T \quad \forall r \in \mathbb{R}$.

4.) $(AB)^T = B^T A^T$

Pf:

§ 2.2 The inverse of a matrix.

- Def: An $n \times n$ matrix A is invertible provided $\exists$ an $n \times n$ matrix C s.t.

$$AC = I_n = CA.$$

  C is ~~...~~ called the inverse of A and is denoted A^{-1}.

  A matrix that is not invertible is said to be singular.

- Def: Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, Then

$$\det(A) = ad - bc$$

- Thm 4:

  Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det(A) \neq 0$. Then

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  If $\det(A) = 0$ then A is singular.

- Pf:

Thm 5: If A is invertible, then $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^n$, namely $\vec{x} = A^{-1}\vec{b}$.

Pf:

Thm 6:

a.) If A is invertible then A^{-1} is inv. and $(A^{-1})^{-1} = A$.

b.) $\cancel{S}$ A, B are invertible $n\times n$ matrices. Then AB is inv. and $(AB)^{-1} = B^{-1}A^{-1}$

c.) If A is inv., then so is A^T and $(A^T)^{-1} = (A^{-1})^T$,

Pf:

~~Def~~: An elementary matrix is one which is obtained by performing a single elementary row op. to I_n.

e.g.:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are elementary.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

Note: 1. $ an elem. row op. is performed to an $m \times n$ matrix A. The resulting matrix can be written as EA where E is the $m \times m$ elem. matrix obtained by performing the same row op. to I_m.

2. Each elem matrix is invertible. E^{-1} is an elem. matrix of the same ~~[illegible]~~ same type of transformation that transforms E back to I_m.

Thm 7:

An $n \times n$ matrix A is inv. iff $A \sim I_n$. In this case, the seq. of elem. row. op.'s that transforms A to I_n also transforms I_n to A^{-1}.

Pf.

eg. Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 3 & 5 & 1 \end{bmatrix}$. Find A^{-1}.