§ 2.3 Characterization of Invertible Matrices

• **Thm 8:** Let A be an $n \times n$ matrix.

T.F.A.E.

1.) A is inv.

2.) $A \sim I_n$

3.) A has n pivots

4.) $A\vec{x} = \vec{0}$ has only the trivial solution.

5.) The col.'s of A are L.I.

6.) $\vec{x} \longmapsto A\vec{x}$ is 1-1.

7.) $A\vec{x} = \vec{b}$ has at least one solution $\forall \vec{b} \in \mathbb{R}^n$.

8.) The col.'s of A span $\mathbb{R}^n$.

9.) $\vec{x} \longmapsto A\vec{x}$ is onto

10.) $\exists$ C s.t. $CA = I$

11.) $\exists$ D s.t. $AD = I$

12.) A^T is inv.

Pf: We will show:

$$1. \Rightarrow 10. \Rightarrow 4. \Rightarrow 3. \Rightarrow 2. \Rightarrow 1.$$

Then we show

$$1. \Rightarrow 11. \Rightarrow 7. \Rightarrow 1., \quad 7 \Leftrightarrow 8 \Leftrightarrow 9$$

$$4 \Leftrightarrow 5 \Leftrightarrow 6, \quad 1. \Leftrightarrow 12.$$

Example:

Is $A = \begin{bmatrix} 0 & 7 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 6 \end{bmatrix}$ inv.?

Theorem 9; Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be linear and let A be its matrix. Then T is inv. iff A is inv. In this case,

$$T^{-1}(\vec{x}) = A^{-1}\vec{x}.$$

Pf:

3.1 Determinants

Determine when the following is inv. using row reduction

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \sim$$

Def: If A is 3×3, The determinant of A is defined by

$$\Delta(A) =$$

- <u>Fact</u>: A is inv. <u>iff</u> $\Delta(A) \neq 0$.

Note: For a 3×3 matrix A,

$$\Delta(A) = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & g \\ f & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Def:

$$A_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & & & \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

remove the i^{th} row and j^{th} col.

• **Def:** If A is $n\times n$. Then

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$

Example:

Compute $\det(A)$ where

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 2 & 0 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

- **Notation:** $\det(A) = |A|$.

- **Def:**

① $C_{ij} = (-1)^{i+j} \det(A_{ij})$

② $|A| = \sum_{j=1}^{n} a_{ij} C_{ij}$ (i is fixed)

③ $|A| = \sum_{i=1}^{n} a_{ij} C_{ij}$ (j is fixed)

- **Example:** Compute $|A|$ where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Thm 2: If A is triangular then
$$|A| = a_{11} a_{22} a_{33} \cdots a_{nn} .$$

Theorem 3: Let A be $n \times n$.

1. If $A \xrightarrow{R_i + cR_j} B$ then $|A| = |B|$.
2. If $A \xrightarrow{R_i \leftrightarrow R_j} B$ then $|A| = -|B|$.
3. If $A \xrightarrow{kR_i} B$ then $B = k|A|$

eg: Compute

$$\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix}$$

Fact: $\exists$ $A \sim U$ where U is in echelon form.

$$|A| = \begin{cases} (-1)^r * (\text{product of pivots in } U), & \text{if } A \text{ is inv.} \\ 0 & \text{o/w} \end{cases}$$

r = # row swaps.

Thm 4: A is inv. iff $|A| \neq 0$.

__Fact:__ If A has 2 rows or 2 col's the same then $|A| = 0$.

__Pf:__

__Note:__ We can exploit the presence of many 0's.

__eg:__ Compute

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 8 & -1 & 5 \\ 1 & 2 & 3 & 4 \end{vmatrix} =$$

Theorem 5:
If A is $n \times n$ then $|A| = |A^T|$.

Pf:

Theorem 6: If A, B are $n \times n$ then
$$|AB| = |A|\,|B|.$$

e.g.: $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

$AB =$

$|A| =$ $|B| =$ $|AB| =$

Pf: (Thm. 6) We will assume Thm. 3.
(i.e. We will assume $|EB| = |E|\,|B|$ if E is elemetary)