§3.3 Cramer's Rule.

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Def: If A is $n \times n$ and $\vec{b} \in \mathbb{R}^n$. Put $A_i(\vec{b}) = [\vec{a}_1 \; \vec{a}_2 \cdots \vec{a}_{i-1} \; \vec{b} \; \vec{a}_{i+1} \cdots \vec{a}_n]$.
(Replace the i^{th} column of A with $\vec{b}$).

Theorem 7 (Cramer's Rule). Let A be an $n \times n$ inv. matrix. For any $\vec{b} \in \mathbb{R}^n$ the unique solution to $A\vec{x} = \vec{b}$ has entries given by

$$x_i = \frac{\det A_i(\vec{b})}{\det A}.$$

Pf:

eg. Use Cramer's rule to solve $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 3 \\ 43 \end{bmatrix}.$$

eg.: Consider the system

$$\begin{cases} 4s x_1 + 2x_2 = 1 \\ 5x_1 + x_2 = -1 \end{cases}$$

For which s is there a unique soln, ?

For such s describe the solution.

- Def: $$\mathrm{Adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & & & \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$= \left(\begin{bmatrix} C_{11} & \cdots & C_{1n} \\ C_{21} & \cdots & C_{2n} \\ \vdots & & \\ C_{n1} & \cdots & C_{nn} \end{bmatrix} \right)^t$$

where $C_{ij} = (-1)^{i+j} |A_{ij}|$.

- Thm 8: Let A be an inv. $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$$

Pf:

eg: Compute A^{-1} where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

- <u>Theorem 9:</u> If A is a 2×2 matrix then the area of the parallelogram determined by its columns is $|\det(A)|$. If A is a 3×3 matrix then the volume of the parallelepiped determined by its columns is $|\det(A)|$.

- <u>Pf:</u>

- <u>Lemma:</u> Let $\vec{a}_1, \vec{a}_2 (, \vec{a}_3)$ be non zero vectors. Then for any $c \in \mathbb{R}$, the area (volume) of the parallelogram (parallelepiped) determined by $\vec{a}_1, \vec{a}_2$ $(\vec{a}_3)$ is the same as the area (volume) determined by $\vec{a}_1, \vec{a}_2 + c\vec{a}_1$ $(\vec{a}_3)$.

- <u>Pf:</u>

- Theorem 10: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be linear with 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\text{Area}(T(S)) = |\det(A)| \cdot \text{Area}(S)$$

If $T: \mathbb{R}^3 \to \mathbb{R}^3$ is linear w/ 3×3 matrix A, and S is a parallelepiped in $\mathbb{R}^3$, then

$$\text{Vol}(T(S)) = |\det(A)| \; \text{Vol}(S).$$

Pf:

- Note: The result of Thm 10 holds for any region S of $\mathbb{R}^2$ or $\mathbb{R}^3$.

Example:

Sp $a, b \in \mathbb{N}$. Find the area bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 25$$