# §4.1 Vector Spaces.



<u>Def.</u> A <u>vector space</u> is a nonempty set $V$ of objects called vectors together with two operations called addition $(+)$ and scalar multiplication $(\cdot)$ satisfying

1.) $u+w \in V \quad \forall u,w \in V$

2.) $u+v = v+u$

3.) $(u+v)+w = u+(v+w)$.

4.) $\exists \ \vec{0} \in V$ s.t. $u+\vec{0} = u \quad \forall u \in V$.

5.) $\forall u \in V \ \exists \ -u \in V$ s.t. $u+(-u) = \vec{0}$.

6.) $cu \in V \quad \forall u \in V, c \in \mathbb{R}$.

7.) $c(u+v) = cu + cv$

8.) $(c+d)u = cu + du$.

9.) $c(du) = (cd)u$

10.) $1 \cdot u = u$

<u>Fact:</u> 1. $0 \cdot u = \vec{0}$

2. $c \cdot \vec{0} = \vec{0}$

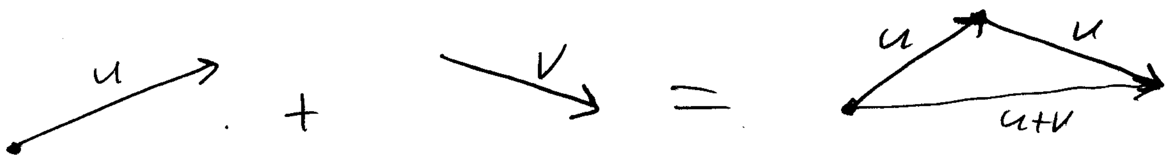
3. $-u = (-1) \cdot u$

<u>Pf:</u>

Eg.: 1. $\mathbb{R}^n$ is a vector space. ($n \geq 1$)

2. Consider arrows $\longrightarrow$ in 3-space which have a length and a direction.

Define $+$ by




$c\vec{u}$ will be the arrow in the same direction as $u$ with length

$c \cdot \text{length}(u)$

(If $c < 0$ we switch the direction)

verify this is a v.s.

**Eg.:** Let $\mathbb{P}_n = \{ f(x) : f \text{ is a polynomial w/ } \deg(f) \le n \}$ bbr

Check that $\mathbb{P}_n$ is a V.S.

**Def:** $H \subseteq V$ is a <u>subspace</u> of $V$ provided

1.) $\vec{0} \in H$.
2.) $u + v \in H \;\; \forall\, u, v \in H$.
3.) $cu \in H \;\; \forall\, u \in H$.

**Eg.:** (1.) $\{\vec{0}\}$ is a subspace of $V$ if $V$ is any V.S.

(2.) If $n < m$ then $\mathbb{P}_n$ is a subspace of $\mathbb{P}_m$

Note: ① $\mathbb{R}^2 \not\subseteq \mathbb{R}^3$.

② A plane through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

Def: ① Suppose $v_1, \dots, v_k \in V$ and $c_1, \dots, c_k \in \mathbb{R}$.
Then

$$\sum_{i=1}^{k} c_i v_i$$

is a linear combination of $v_1, \dots, v_k$ w/ weights $c_1, \dots, c_k$.

② $\text{span}(v_1, \dots, v_k)$ denotes the set of all linear combinations of $v_1, \dots, v_k$.

Thm. 1 If $v_1, \dots, v_k \in V$, then $\text{span}(v_1, \dots, v_k)$ is a subspace of $V$.

Pf:

## §4.2 Null Spaces, Column Spaces & Linear Transformations.



- **Def:** We define the null space of a matrix $A$ by
$$Nul(A) = \{\vec{x} ; A\vec{x} = \vec{0}\}$$
- **Thm 2:** If $A$ is $m \times n$ then $Nul(A)$ is a subspace of $\mathbb{R}^n$.

Pf:

- **eg.** Let $H = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a - 2b + 5c = d ; c - a = b \right\}$

Show that $H$ is a subspace of $\mathbb{R}^4$

- eg: Find a spanning set for the null space of 

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

- Note: 1. The spanning set produced by the above method is automatocally indep.

2. when $Nul(A) \neq \{\vec{0}\}$, the # of nonzero vectors in a spanning set of $Nul(A)$ is equal to the # of free variables.

- **Def:** Suppose $A = [\vec{a}_1, \dots, \vec{a}_n]$ then
$$\mathrm{Col}(A) := \mathrm{Span}(\vec{a}_1, \dots, \vec{a}_n)$$

- **Thm:** $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$ where $A$ is $m \times n$.

- **Pf:** Suppose $A$ is $m \times n$.
Note: $\mathrm{Col}(A) = \{ A\vec{x} : \vec{x} \in \mathbb{R}^n \}$.

- **eg:** Find $A$ s.t. $\mathrm{Col}(A) = \left\{ \begin{bmatrix} 5a-b \\ 3b+2a \\ -7a \end{bmatrix} : a, b \in \mathbb{R} \right\}$.

- **Note:** For $A$ an $m \times n$ matrix $\mathrm{Col}(A) = \mathbb{R}^m$
iff $A\vec{x} = \vec{b}$ has a soln. $\forall\, \vec{b} \in \mathbb{R}^m$.
iff The linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ with matrix $A$ is onto.

Example: Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$

1) $\mathrm{Col}(A) < \underline{\qquad\qquad}$. ($<$ means "is a subspace of")

2) $\mathrm{Nul}(A) < \underline{\qquad\qquad}$.

3) Give any vector in $\mathrm{Col}(A)$.

4) Describe $\mathrm{Nul}(A)$ in vector parametric form.

$$A \sim \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

5) Let $\vec{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$

Is $\vec{u}$ and/or $\vec{v}$ in $\mathrm{Col}(A)/\mathrm{Nul}(A)$?

- **Def:** $V$ and $W$ are vector spaces. A transformation $T: V \to W$ is said to be linear provided that
  1) $T(u+v) = T(u) + T(v) \quad \forall u, v \in V.$
  2) $T(\alpha u) = \alpha T(v) \quad \forall \alpha \in \mathbb{R}, u \in V.$

- **Def:** We define the kernel and range of $T: V \to W$ are defined as

$$\ker(T) := \{ v \in V : T(v) = 0 \}$$

$$\operatorname{range}(T) = \operatorname{im}(T) := \{ T(v) : v \in V \}.$$

- **Fact:** Given a linear transformation $T: V \to W$, ~~ker(T) and im(T)~~
  1) $\ker(T) < V$
  2) $\operatorname{im}(T) < W.$

- **Pf:**