§ 4.3 Linearly Independent Sets; Bases

Def. If V is a vector space and $v_1, \ldots, v_p \in V$. The set $\{v_1, \ldots, v_p\}$ is <u>Linearly Independent</u> provided that the eq.

$$x_1 v_1 + \ldots + x_p v_p = 0 \quad (*)$$

has only the trivial solution.

If the equation $(*)$ has non trivial solutions then $\{v_1, \ldots, v_p\}$ is <u>linearly dependent</u>.

Theorem 4: If $S = \{v_1, \ldots, v_p\} \subseteq V$; $p \geq 2$; $v_1 \neq 0$. Then S is L.D. <u>iff</u> there is some v_j which is a linear combination of the preceeding vectors $v_1, v_2, \ldots, v_{j-1}$

Pf.

Example:

Let $S = \{ x^2+2x+3,\ x^3+1,\ x^3+2x^2+4x+7 \}$
$\subseteq \mathbb{P}_3$

Is S L.I. or L.D. ?

Example: What about

$S = \{ \sin(x), \cos(x) \} \subseteq \left\{ f : [0,1] \to [0,1] : \begin{matrix} \\ f \text{ is cont.} \end{matrix} \right\}$?

- Def: $\mathcal{F}$ $H < V$. an indexed set of vectors $\mathcal{B} = \{ b_1, \dots, b_p \}$ in V is a basis for H provided that
  1.) $\mathcal{B}$ is L.I.
  2.) $H = \operatorname{span}(\mathcal{B})$.
- Note: H could be all of V.
- Example: Let $A = [\vec{a}_1, \dots, \vec{a}_n]$ be an $n \times n$ invertible matrix. What does the Inv. Matrix Theorem say about $\{ \vec{a}_1, \dots, \vec{a}_n \}$?

Example: $\{e_1, \dots, e_n\}$ is a basis for $\mathbb{R}^n$. Why?

Example: $\{1, x, x^2, \dots, x^n\}$ is a basis for $\mathbb{P}_n$. Why?

Example:

$\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

Show that $\text{Span}(v_1, v_2, v_3) = \text{Span}(v_1, v_2)$

Theorem 5: Let $S = \{v_1, \dots, v_p\} \subseteq V$; $H = \text{Span}(v_1, \dots, v_p)$.

a) If $\exists\, v_k \in S$ which is a linear comb. of other vectors in S, then
$$H = \text{Span}(v_1, v_2, \dots, v_{k-1}, v_{k+1}, \dots, v_p)$$

b) If $H \neq \{0\}$ then some subset of S is a basis for H.

Pf:

Bases for Nul(A), Col(A):

- Note: We already know how to find bases for Nul(A) from the last section.

Example: Let

$$B = \begin{bmatrix} 1 & 3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for Col(B).

Fact: If $A \sim B$ then the columns of A have exactly the same dependencies as those of B.

Pf:

Example:

Let $A = \begin{bmatrix} 1 & 3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 1 & 5 & 1 \\ 1 & 3 & 1 & 7 & 1 \end{bmatrix}$

Find a basis for $Col(A)$.

Thm 6: The pivot columns of A form a basis for $\text{Col}(A)$.

Pf:

Note: A basis is...

1) A spanning set that is as small as possible.

2) A L.I. set that is as big as possible.

Example: which are bases for $\mathbb{R}^3$?

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$