§4.4 Coordinate Systems.

Theorem 7: (Unique Representation Theorem)
Let $\mathcal{B} = \{b_1, \dots, b_n\}$ be a basis for a vector space V. For all $\vec{v} \in V$, $\exists!$ $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ s.t.

$$\vec{v} = x_1 b_1 + \dots + x_n b_n$$

- Pf: Since $\mathcal{B}$ spans V, we have existence of such scalars $x_1, \dots, x_n$.
Now suppose that we also have

$$\vec{v} = y_1 b_1 + \dots + y_n b_n$$

$$\Rightarrow \quad \sum_{i=1}^{n} (x_i - y_i) b_i = 0$$

$$\Rightarrow \quad x_i - y_i = 0 \ \forall i \quad (\because \mathcal{B} \text{ is L.I.})$$

$$\Rightarrow \quad x_i = y_i \quad \text{for } i = 1, \dots, n \quad \blacksquare$$

• Notation: $\mathcal{B}$ and V are as above.
Given $\vec{v} \in V$, we write

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

provided $\vec{v} = \sum x_i b_i$.

Example:

$\$$ $b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $b_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

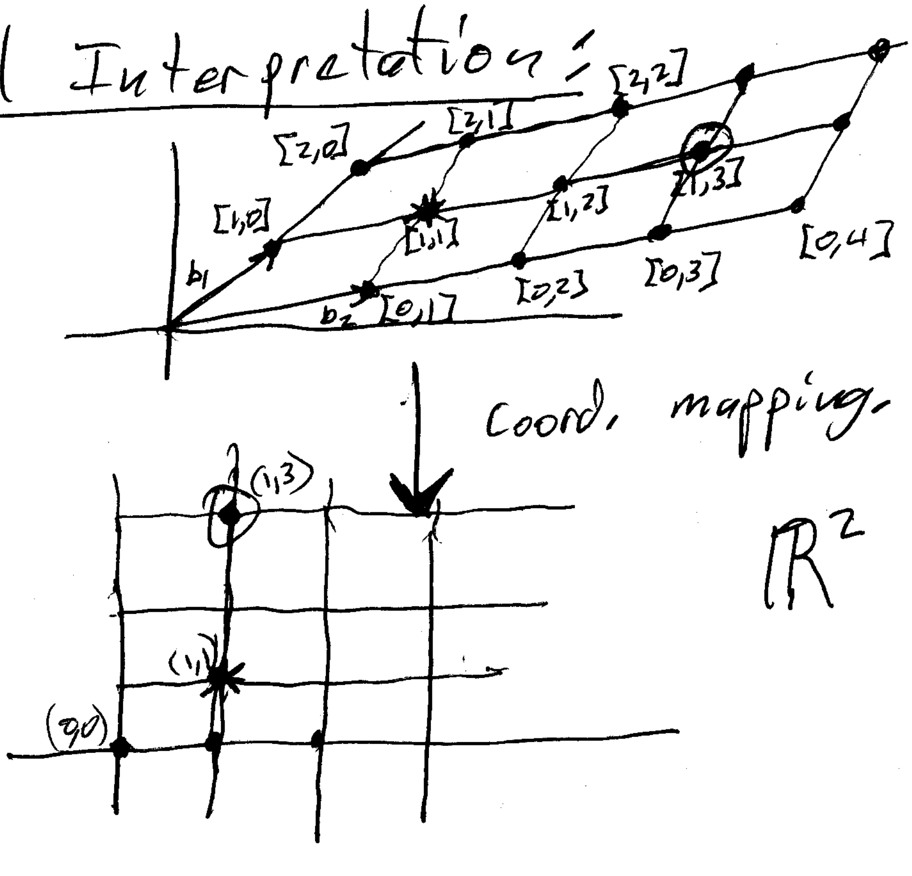
Then $\mathcal{B} = \{b_1, b_2\}$ is a basis for $\mathbb{R}^2$,

$\$$
$$[x]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$$

what is $\vec{x}$? $\vec{x} =$

$\$$ $\vec{x} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$. What is $[\vec{x}]_{\mathcal{B}}$?

- Graphical Interpretation:



Coordinates in $\mathbb{R}^n$

Suppose that we are given a basis $\mathcal{B} = \{b_1, \dots, b_n\}$ for $\mathbb{R}^n$

Then the vector eq.

$$\vec{v} = c_1 b_1 + \dots + c_n b_n$$

is equivalent to

$$\vec{v} = \underbrace{\begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{bmatrix}}_{P_{\mathcal{B}}} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

so, we have

$$\vec{v} = P_{\mathcal{B}} \, [\vec{v}]_{\mathcal{B}} .$$

Definition: $P_{\mathcal{B}}$ is called the change of coordinates matrix.

(It changes from $\mathcal{B}$-coord.'s to standard coord.'s).

Note:

$$P_{\mathcal{B}}^{-1} \vec{v} = [\vec{v}]_{\mathcal{B}}$$

Theorem 8: Suppose that $\mathcal{B} = \{b_1, \ldots, b_n\}$ is a basis for a vector space V. Let $T: V \to \mathbb{R}^n$ be defined by

$$T(\vec{v}) = [\vec{v}]_{\mathcal{B}}.$$

Then T is a 1-1 linear mapping.

Pf:

Cor: (1) $[\vec{v}]_{\mathcal{B}} = \vec{0} \iff \vec{v} = \vec{0}$.

(2) $[\underbrace{c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n}_{\in V}]_{\mathcal{B}} = c_1 [v_1]_{\mathcal{B}} + \cdots + c_n [v_n]_{\mathcal{B}}.$

eg.: $\mathbb{P}_3$ has basis $\mathcal{B} = \{1, x, x^2, x^3\}$

If $p(x) = ax^3 + bx^2 + cx + d \in \mathbb{P}_3$

then $[p(x)]_{\mathcal{B}} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$.

Thus $\mathbb{P}_3 \cong$ ________ .

eg.:

Consider $S = \left\{ \begin{array}{l} p(x) = 1 + x + x^3,\ q(x) = 2 + x^2, \\ r(x) = 4 + 2x + x^2 + 2x^3,\ s(x) = 1 + x + x^2 + x^3 \end{array} \right\}$

Is S L.I. or L.D.?

Example:

$\mathcal{S}$ $v_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$; $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$; $\vec{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$.

Then $\mathcal{B} = \{v_1, v_2\}$ is a basis for $H = \text{Span}(v_1, v_2)$

Is $\vec{x} \in H$ and if so what is $[\vec{x}]_{\mathcal{B}}$?