§4.5 The dimension of a vector space.

- **Theorem 9:** If a v.s. V has a basis $\mathcal{B} = \{b_1, \dots, b_n\}$ then any set in V containing more than n vectors is L.D.

- Pf:

- **Theorem 10:** If a v.s. V has a basis $\mathcal{B} = \{b_1, \dots, b_n\}$ then every basis of V has exactly n vectors.

- Pf:

- **Definition:** (1.) If V is spanned by a finite set then V is finite dimensional and $\dim(V) := \#$ vectors in any basis for V.

  (2.) Convention: $\dim(\{0\}) = 0$; basis for $\{0\}$ is $\emptyset$.

  (3.) If V is not spanned by any finite subset of V, then V is infinite dimensional $(\dim(V) = \infty)$.

Example:

$$\dim(\mathbb{R}^4) =$$

$$\dim(\mathbb{P}_n) =$$

$$\dim(\mathbb{P}) =$$

Example: Find the dimension of

$$H = \left\{ \begin{bmatrix} a + 4b + c + 2d \\ a + 2b + d \\ a + 5b + c + 3d \\ b + d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

- **Theorem 11:**

  If $H \leq V$ and V is a f.d.v.s. Any LI set $S \subseteq H$ can be expanded to a basis for H and

  $$\dim(H) \leq \dim(V).$$

- **Pf**

- **Theorem 12:** Let V be a p-dimensional v.s. with $p \geq 1$. (1) Any LI set of exactly p vectors in V is a basis for V.

  (2) Any set of p vectors of V which spans V is a basis for V.

Pf:

Dimensions of Nul(A) & Col(A).

dim(Col A) = # pivots in A.

dim(Nul A) = # free variables in A
(= # col.'s in A — # pivots).

- **Example**:
Let $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & 4 & 5 & 8 & 4 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

- what is dim(Nul(A)) & dim(Col(A))?

- **Example**: Subspaces of $\mathbb{R}^3$.
dim(V) = 0
dim(V) = 1
dim(V) = 2
dim(V) = 3