§4.6 Rank.

- **Def:** Suppose that $A = \begin{pmatrix} - r_1 - \\ - r_2 - \\ \vdots \\ - r_m - \end{pmatrix}$

  Then $\mathrm{Row}(A) = \mathrm{Span}(r_1, \dots, r_m)$
  (i.e. $\mathrm{Row}(A) = \mathrm{Col}(A^t)$).

- **Thm B:** If $A \sim B$ then $\mathrm{Row}(A) = \mathrm{Row}(B)$. If B is in echelon form then the nonzero rows of B are a basis for $\mathrm{Row}(B) = \mathrm{Row}(A)$.

- **Pf:**

- Exercise: Let $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$.

Find bases for $Row(A)$, $Col(A)$, $Nul(A)$.

Note: $A \sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

- Theorem (The Rank-Nullity Theorem).

S A is $m \times n$. Then

(I) $\dim(Row(A)) = \dim(Col(A))$

We call this value $rank(A)$ and further we have

$$rank(A) + \dim(Nul(A)) = n$$

- Pf:

Theorem: Let A be $n \times n$. The following are equivalent to A being invertible.

(m) The columns of A are a basis for $\mathbb{R}^n$.

(n) $\text{Col}(A) = \mathbb{R}^n$.

(o) $\dim(\text{Col}(A)) = n$.

(p) $\text{rank}(A) = n$

(q) $\text{Nul}(A) = \{\vec{0}\}$

(r) $\dim(\text{Nul}(A)) = 0$.