§ 4.7 Change of Basis.

Example: Suppose we have two bases $\mathcal{B}$ and $\mathcal{C}$ for a vector space V. Suppose

$$b_1 = 4c_1 + 3c_2$$

$$b_2 = 2c_1 + c_2$$

Suppose $[x]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Find $[x]_{\mathcal{C}}$.

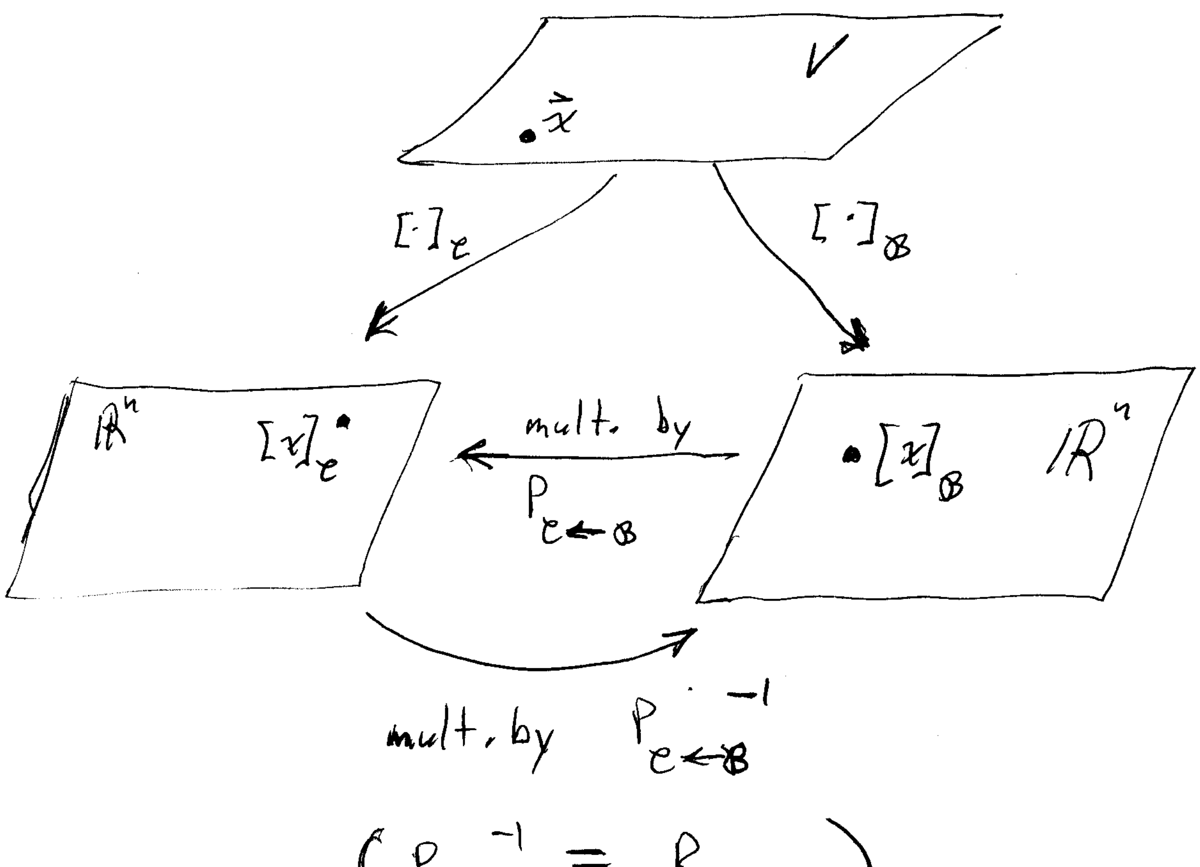
Theorem:

Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ and $\mathcal{C} = \{c_1, \ldots, c_n\}$ be bases for a vector space V. Then there is a unique $n \times n$ matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ such that

$$[x]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}} [x]_{\mathcal{B}}$$

The columns of $P_{\mathcal{C} \leftarrow \mathcal{B}}$ are the $\mathcal{C}$-coordinate vectors of the vectors in $\mathcal{B}$.

i.e. $P_{\mathcal{C} \leftarrow \mathcal{B}} = \left[[b_1]_{\mathcal{C}} \; \cdots \; [b_n]_{\mathcal{C}} \right]$



$$\left(P_{\mathcal{C} \leftarrow \mathcal{B}}^{-1} = P_{\mathcal{B} \leftarrow \mathcal{C}} \right)$$

Example:

Let $b_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$ $b_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$

$c_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ $c_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

Let $\mathcal{B} = \{b_1, b_2\}$ & $\mathcal{C} = \{c_1, c_2\}$ be bases for $\mathbb{R}^2$. Find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

4.7-4

By Thm 15,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \left[[b_1]_{\mathcal{C}} \ [b_2]_{\mathcal{C}} \right]$$

So, we need to find

$$[b_1]_{\mathcal{C}} \quad \& \quad [b_2]_{\mathcal{C}}$$

i.e. solve

$$b_1 = p_{11} c_1 + p_{21} c_2 = \begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix}$$

$$b_2 = p_{12} c_1 + p_{22} c_2$$

To solve these 2 vector eq's simultaneously consider

$$\begin{array}{c} \quad\quad\quad\quad b_1 \quad\quad \\ \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right] \end{array}$$

$$\to \begin{bmatrix} 1 & 3 & -9 & -5 \\ 0 & 7 & -35 & -21 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -9 & -5 \\ 0 & 1 & -5 & -3 \end{bmatrix}$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right]$$

so,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$

Note:

$$\left[\begin{array}{cc:cc} | & | & | & | \\ c_1 & c_2 & b_1 & b_2 \\ | & | & | & | \end{array}\right] \longrightarrow \left[\begin{array}{c:c} I & P_{\mathcal{C}\leftarrow\mathcal{B}} \end{array}\right]$$

Recall: For $x \in \mathbb{R}^n$ we have

$$P_{\mathcal{B}}[x]_{\mathcal{B}} = x \qquad P_{\mathcal{B}} = [b_1, \dots, b_n]$$

$$P_{\mathcal{C}}[x]_{\mathcal{C}} = x \qquad P_{\mathcal{C}} = [c_1, \dots, c_n]$$

So,

$$[x]_{\mathcal{C}} = P_{\mathcal{C}}^{-1} x = P_{\mathcal{C}}^{-1} P_{\mathcal{B}} [x]_{\mathcal{B}}$$

So,

$$\boxed{P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}}^{-1} P_{\mathcal{B}}}$$

However $P_{\mathcal{C}\leftarrow\mathcal{B}}$ can be computed more quickly as in the previous example.