§5.1 Eigenvectors & Eigenvalues.

- **Definition:**
An *eigenvector* of an $n \times n$ matrix A is a nonzero vector $\vec{v}$ such that

$$A\vec{v} = \lambda \vec{v}$$

for some scalar λ which is called an eigenvalue of A.

- **Example:**

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

So, -4 is an eigenvalue of $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$ is an eigenvector.

- **Example:** Show that 7 is an eigenvalue for $A = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$.

- **Note:** For any scalar λ,

$$A\vec{x} = \lambda\vec{x}$$
$$\Leftrightarrow (A - \lambda I)\vec{x} = \vec{0}$$
$$\Leftrightarrow \vec{x} \in Nul(A - \lambda I).$$

- **Def:** If $\dim(Nul(A - \lambda I)) > 0$, then $Nul(A-\lambda I)$ is called the eigenspace for A corresponding to λ, since any $\vec{v} \in Nul(A - \lambda I)$ satisfies

$$A\vec{v} = \lambda\vec{v}.$$

- **Example:**
Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$

Find a basis for the eigenspace corresponding to $\lambda = 2$.

- Theorem 1:

The eigenvalues of a triangular matrix are its diagonal entries.

Pf:

- Note: $\vec{0}$ is an eigen value of A

$\Leftrightarrow \exists \vec{x} \neq \vec{0}$ s.t. $A\vec{x} = 0 \cdot \vec{x} = \vec{0}$

$\Leftrightarrow \exists \vec{x} \neq 0 \in Nul(A)$

$\Leftrightarrow A$ is not inv.

- Theorem 2: If $v_1, v_2, \ldots, v_r$ are eigenvectors corresponding to distinct eigen values $\lambda_1, \ldots, \lambda_r$ then $\{v_1, \ldots, v_r\}$ is L.I.

- Pf:

<u>Example</u>: § we would like a sequence of vectors $x_0, x_1, x_2, \ldots$ such that

$$x_{k+1} = A x_k$$

By solution, we mean a description of the x_k which does not depend on A.

Then simplest solution is to take x_0 to be an eigenvector of A w.r.t. some eigenvalue λ.

Then,

$$\begin{aligned} x_1 &= A x_0 = \lambda x_0 \\ x_2 &= A x_1 = \lambda x_1 = \lambda^2 x_0 \\ &\vdots \\ x_k &= \lambda^k x_0 \end{aligned}$$

§5.2 The Characteristic Equation

Example:

Let $A = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$.

Find the eigenvalues of A.

Determinant Review

- **Fact:** If A is $n\times n$ and $A \sim U$ where U is upper triangular then

$$|A| = (-1)^r \prod_{i=1}^{n} u_{ii}$$

where $r = \#$ row interchanges to get from A to U.

- **Note:** $\exists\ 1 \le i \le n$ s.t. $u_{ii} = 0 \iff A$ is singular $\iff |A| = 0$.

- **Theorem: (Inv. Matrix Thm (cont.)).**

Let A be $n\times n$. Then A is inv. iff

(s) 0 is not an eigen value of A.

(t) $|A| \neq 0$.

- **Thm 3 (Properties of Determinants).**

Let A, B be $n\times n$ matrices.

(1) A is inv. iff $|A| \neq 0$.

(2) $|AB| = |A|\,|B|$

(3) $|A^t| = |A|$

(4) If A is triangular then
$$|A| = \prod_{i=1}^{n} a_{ii}$$

(5) (i) Row replacement does not change the determinant.
(ii) Row interchange changes the sign of the determinant.
(iii) Scaling a row, scales the determinant by the same factor.

Characteristic Eq'n

Fact: λ is an eigenvalue of A iff it satisfies the characteristic equation

$$|A - \lambda I| = 0$$

degree n polynomial in λ

Pf:

Def: The multiplicity of the eigenvalue λ is the power of $(x - \lambda)$ in the factorization of the characteristic equation for A

- **Example:** Find the eigenvalues and their multiplicities of

$$A = \begin{bmatrix} 6 & 5 & 0 & -5 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

Similarity:

- Def: A is similar to B ~~iff~~ provided $\exists$ an inv. matrix P s.t.

$$A = PBP^{-1}$$

$$(\Leftrightarrow \quad P^{-1}AP = B).$$

Theorem 4: If A & B are similar, then they have the same characteristic polynomial and thus the same eigenvalues with the same mult.

Pf

Example:

Let $A = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}$

Take $x_0 = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$ and $x_{k+1} = Ax_k \quad k = 0, 1, \ldots$

Analyze the long term behavior of the system.