§5.3 Diagonalization

Definition: $D = (d_{ij})_{\substack{1\le i\le n\\ 1\le j\le n}}$ is diagonal provided that when $i \neq j$, $d_{ij} = 0$.

Example: $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is diagonal.

compute D^k for $k = 1, 2, 3, \ldots$

$$D^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} =$$

$$D^3 = D^2 \cdot D =$$

$$\vdots$$

$$D^k =$$

Note: Given A, if we can find an inv. matrix P and a diagonal matrix D s.t. $A = PDP^{-1}$

Then

$$A^2 =$$
$$A^3 =$$
$$\vdots$$
$$A^k =$$

Definition: An $n \times n$ matrix is said to be **diagonalizable** provided $\exists$ D, diagonal with $A \approx D$ (i.e. $A = PDP^{-1}$, some P)

Theorem 5:
An $n \times n$ matrix is diagonalizable iff it has n L.I. eigenvectors. In fact, when A has n L.I. eigenvectors $v_1, \dots, v_n$, we have

$$A = PDP^{-1}$$

where $P = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$ and

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}.$$

Pf:

Note: If A has n L.I. eigenvectors, then they form a basis for $\mathbb{R}^n$ called an <u>eigen vector basis</u>. 110

Example: Diagonalize

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

<u>Theorem 6:</u> $\mathcal{S}$ A is $n \times n$ and has n distinct eigenvalues. Then A is diagonalizable.

<u>Pf:</u>

<u>Theorem 7:</u> Let A be $n \times n$ w/ distinct eigenvalues $\lambda_1, \ldots, \lambda_p$.

(a.) For $1 \leq k \leq p$, the dimension of the eigenspace w.r.t. λ_k is $\leq$ mult. of λ_k (i.e. $\dim(\mathrm{Nul}(A - \lambda_k I)) \leq \mathrm{mult.}(\lambda_k)$).

(b.) A is diagonalizable <u>iff</u>

$$\sum_{k=1}^{p} \dim(\mathrm{Nul}(A - \lambda_k I)) = n$$

$$\iff \dim(\mathrm{Nul}(A - \lambda_k I)) = \mathrm{mult.}(\lambda_k) \text{ for } 1 \leq k \leq p.$$

(c.) If A is diagonalizable and if $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p$ are bases for the eigenspaces of A then $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_p$ is an eigenvector basis for $\mathbb{R}^n$.