§ 6.1 Inner Product, Length, Orthogonality.

Def: Inner Product (or Dot Product)

$$u \cdot v = u^t v$$

Note:

If $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$; $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ then

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

Example:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} =$$

Theorem 1: $\forall$ $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$; $c \in \mathbb{R}$. Then

1.) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

2.) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

3.) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = u \cdot (c\vec{v})$

4.) $\vec{u} \cdot \vec{u} \geq 0$ & $\vec{u} \cdot \vec{u} = 0$ iff $\vec{u} = \vec{0}$.

5.) $(c_1\vec{u}_1 + \dots + c_n\vec{u}_n) \cdot \vec{w} = c_1(\vec{u}_1 \cdot \vec{w}) + \dots + c_n(\vec{u}_n \cdot \vec{w})$

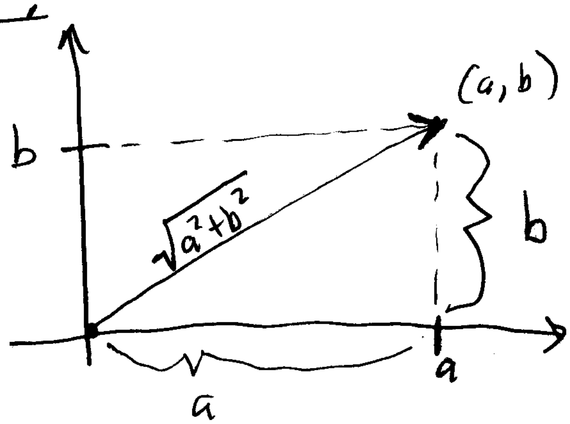
Length: The length of $\vec{v} \in \mathbb{R}^n$ is

$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

And

$$\vec{v}\cdot\vec{v} = \|\vec{v}\|^2$$

• **$\mathbb{R}^2$:**



• **Fact:** $\|c\vec{v}\| = |c|\,\|\vec{v}\|$ $\quad (c \in \mathbb{R},\ \vec{v} \in \mathbb{R}^n)$

• **Unit vector:** $\vec{u}$ is a unit vector if $\|\vec{u}\| = 1$.

• **Normalizing:** Normalizing a vector $\vec{v}$ simply means to replace $\vec{v}$ with $\frac{1}{\|\vec{v}\|}\vec{v}$ which is a unit vector pointing in the same direction as $\vec{v}$.

(Note: $\left\| \frac{1}{\|\vec{v}\|}\vec{v} \right\| = \frac{1}{\|\vec{v}\|}\|\vec{v}\| = 1$)

• **Example:** Find a unit vector which points in the same direction as $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

$\vec{u} =$

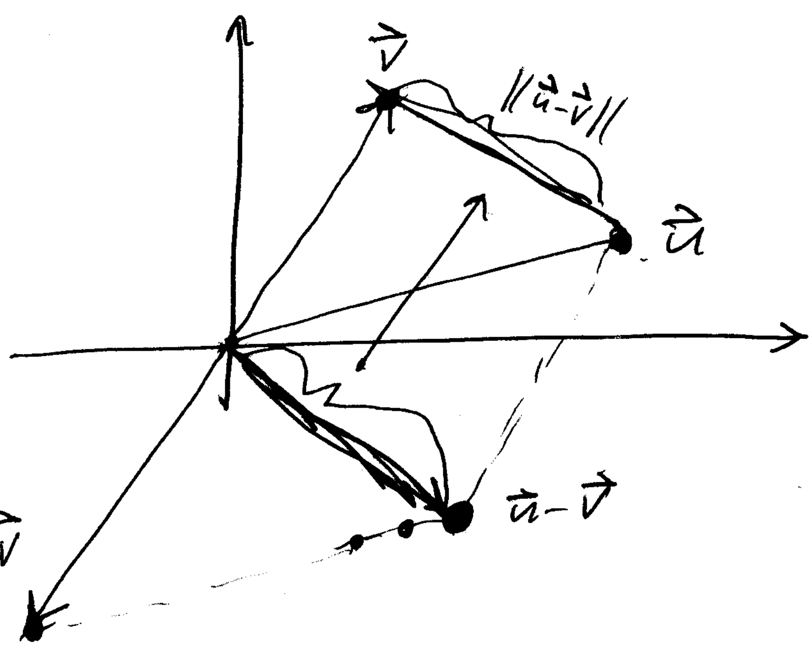
- **Distance:** For $u, v \in \mathbb{R}^n$

$$\text{dist}(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$$

- **e.g.** Compute $\text{dist}\left(\begin{bmatrix}7\\2\end{bmatrix}, \begin{bmatrix}4\\3\end{bmatrix}\right)$

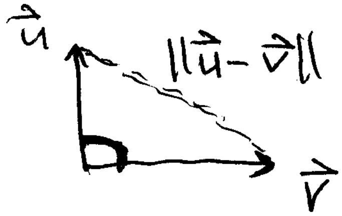
$$= \left\| \begin{bmatrix}7\\2\end{bmatrix} - \begin{bmatrix}4\\3\end{bmatrix} \right\| = \left\| \begin{bmatrix}3\\-1\end{bmatrix} \right\|$$

$$= \sqrt{3^2+(-1)^2} = \sqrt{10}$$



$$\vec{v} + (\vec{u}-\vec{v}) = \vec{u}$$

- Orthogonal Vectors.



$$(\text{dist}(\vec{u},\vec{v}))^2 = \|\vec{u}-\vec{v}\|^2 = (u_1-v_1)^2 + \cdots + (u_n-v_n)^2$$

$$= (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$

$$= (\vec{u}-\vec{v})\cdot\vec{u} - (\vec{u}-\vec{v})\cdot\vec{v}$$

$$= \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\,\vec{u}\cdot\vec{v}$$

- Pythagorean Theorem: If $\vec{u} \perp \vec{v}$ then

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

Thus $\vec{u}\cdot\vec{v} = 0$

- Def: [redacted] $\vec{u}, \vec{v} \in \mathbb{R}^n$ are orthogonal if

$$\vec{u}\cdot\vec{v} = 0$$

- Theorem 2: (Pythagorean Thm.)

$$\vec{u} \perp \vec{v} \quad \text{iff} \quad \|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

- **Def:** (Orthogonal Complements)

  $\$$ $W < \mathbb{R}^n$.

  1.) If $\forall \vec{w} \in W$, $\vec{z} \cdot \vec{w} = 0$ then

  $\vec{z} \perp W$

  ($\vec{z}$ is orthogonal to W)

  2.) $W^\perp := \{ \vec{z} \in \mathbb{R}^n : z \perp W \}$

  is the orthogonal complement of W.

- **Example:** $\$$ $W = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$

  Describe $W^\perp$.

- **Facts:**

1.) $\vec{x} \in W^\perp$ iff $\vec{x} \perp \vec{w}$ $\forall$ $\vec{w}$ is some spanning set for W.

2.) $W^\perp < \mathbb{R}^n$ ($W^\perp$ is a subspace of $\mathbb{R}^n$)

Theorem 3: S A is an $m \times n$ matrix.

1.) $(\text{Row}(A))^\perp = \text{Nul}(A)$

2.) $(\text{Col}(A))^\perp = \text{Nul}(A^T)$

- **Fact:** If $\vec{u}, \vec{v} \in \mathbb{R}^2$ (or $\mathbb{R}^3$) then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta)$$