§6.2 Orthogonal sets.

- **Definition:**
  a set $\{u_1, \ldots, u_k\}$ is <u>orthogonal</u> if
  $$u_i u_j = 0 \qquad \forall i \neq j$$

- **Example:**
  Show that $\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -2 \\ -7/2 \end{bmatrix} \right\}$
  is orthogonal.

- **Theorem 4:** If $S = \{u_1, \ldots, u_p\}$ is an orthogonal set in $\mathbb{R}^n$, then S is L.I. and hence is a basis for $\text{Span}(S)$.

- **Pf**

- Definition:

An <u>orthogonal basis</u> for $W < \mathbb{R}^n$ is a basis for W which is orthogonal.

- <u>Theorem 5:</u>

Let $\{u_1, \ldots, u_p\}$ be an orthogonal basis for $W < \mathbb{R}^n$. Then for each $y \in W$, we have

$$y = c_1 u_1 + \cdots + c_p u_p$$

where

$$c_j = \frac{y \cdot u_j}{u_j \cdot u_j}$$

- <u>Pf:</u>

<u>Example:</u> Note that $S = \left\{ \begin{array}{ccc} u_1 & u_2 & u_3 \\ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, & \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, & \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix} \end{array} \right\}$

is an ortho basis for $\mathbb{R}^3$. Express $y = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ as a lin comb. of u_1, u_2 & u_3.

Orthogonal Projections:

Given $y \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$, find $\hat{y}, z \in \mathbb{R}^n$ such that

1.) $y = \hat{y} + z$

2.) $\hat{y} = \alpha u \quad (\alpha \in \mathbb{R})$

3.) $u \cdot z = 0$

• Notation: $\hat{y} = \text{Proj}_L(y) = \left(\frac{y \cdot u}{u \cdot u}\right) u$

where $L = \text{Span}(u)$.

Terminology:

$$y = \hat{y} + z$$

$\hat{y}$: ortho. projection of y onto L.

z: component of y orthogonal to u.

• <u>Example:</u>

Let $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$, $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $L = \text{Span}(u)$.

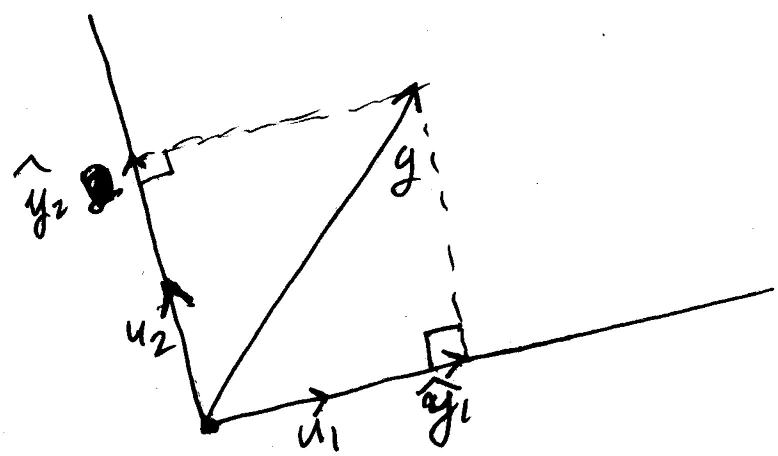
Compute the ortho. proj. of y onto L and the component of y ortho. to L.
Plot y, u, $\hat{y}$ and z. Compute the distance from y to L.

Geometric Interpretation of Thm 5.

$\exists$ $\{u_1, u_2\}$ is an ~~basis~~ ortho basis for $\mathbb{R}^2$

Put
$$\hat{y}_1 = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 = \text{Proj}_{u_1}(y)$$
$$\hat{y}_2 = \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \text{Proj}_{u_2}(y)$$

Then $y = \hat{y}_1 + \hat{y}_2$.



Orthonormal Sets.

__Def:__ A set $\{u_1, \ldots, u_p\}$ is __orthonormal__ if it is orthogonal and $\|u_i\| = 1$ for $1 \le i \le p$. If $W = \text{span}(u_1, \ldots, u_p)$ then the set is an __orthonormal basis__ for W.

- __Example:__ $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $\mathbb{R}^n$.

- __Example:__ Show that $\{v_1, v_2, v_3\}$ is an orthonormal basis for $\mathbb{R}^3$ where

$$v_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix} \quad v_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \quad v_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$$

- Theorem 6:
  An $m \times n$ matrix U has orthonormal columns iff $U^T U = I$
- Pf:

- Theorem 7: If U is $m \times n$ w/ orthonormal columns and $x, y \in \mathbb{R}^n$, Then
  1. $\|Ux\| = \|x\|$
  2. $(Ux) \cdot (Uy) = x \cdot y$
  3. $(Ux) \cdot (Uy) = 0$ iff $x \cdot y = 0$

Note: a. U preserves length
b. U preserves orthogonality.

- Pf:

- Example: Let

$$U = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \quad x = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$$

Note that U has orthonormal columns and that $\|Ux\| = \|x\|$

- Example:

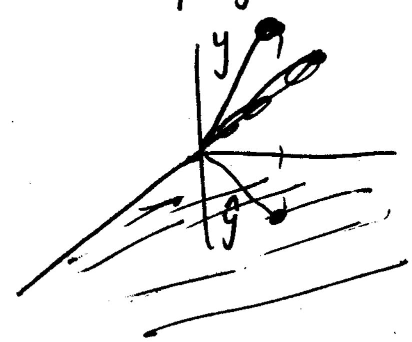
Let $U = \begin{bmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \\ 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \\ 1/\sqrt{11} & 1/\sqrt{6} & 7/\sqrt{66} \end{bmatrix}$

We have seen that the columns are orthonormal. Verify that the rows are orthonormal also.

6.3 Orthogonal Projections

We can extend the ideas in section 6.2 of ~~vectors~~ vector projections onto vectors and lines to vector projections onto subspaces.

* Canonical example: x, y, z axes ($\mathbb{R}^3$), vector projected onto xy-plane

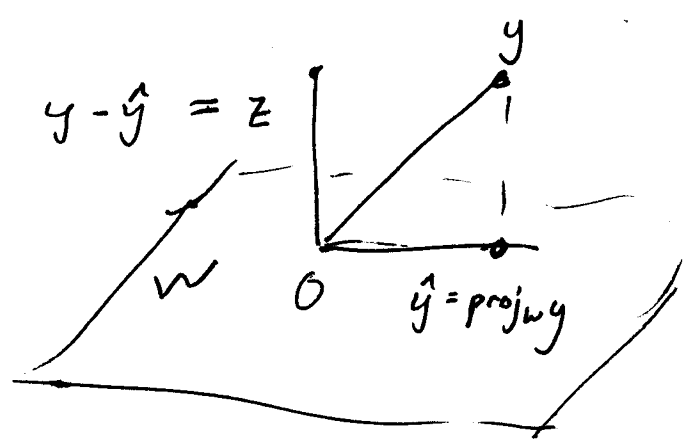


Theorem 8: (Orthogonal Decomposition Theorem)

Let W be a subspace of $\mathbb{R}^n$. Then each $\bar{y} \in \mathbb{R}^n$ can be written <u>uniquely</u> in the form $y = \hat{y} + z$ where $\hat{y} \in W$, $z \in W^{\perp}$.

If $\{u_1, \dots, u_p\}$ is any orthogonal basis of W, then

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \dots + \frac{y \cdot u_p}{u_p \cdot u_p} u_p, \text{ and } z = y - \hat{y}.$$



Example:

Let $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Note that $u_1 \perp u_2$. Let $W = \text{span}(u_1, u_2)$

Write $y = \hat{y} + z$ where $\hat{y} \in W$; $z \in W^{\perp}$.

Notation: $\hat{y} = \text{Proj}_W(y)$.

Thm 9: (Best Approximation Theorem)

Let $W < \mathbb{R}^n$; $y \in \mathbb{R}^n$; $\hat{y} = \text{Proj}_W(y)$.

Then $\hat{y}$ is the closest point in W to y in the sense that

$$\|y - \hat{y}\| < \|y - v\| \qquad \forall v \in W, v \neq \hat{y}.$$

Pf:

Example:

Let $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$; $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, ; $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$;

$W = \text{Span}(u_1, u_2)$ as before.

Then the closest point to y in W is $\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \begin{bmatrix} \end{bmatrix}$.

Example:

Let $u_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $y = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}$,

$W = \text{Span}(u_1, u_2)$.

Compute $\text{dist}(y, W) := \text{dist}(y, \hat{y})$.

Theorem 10: $\mathcal{S}$ $\{u_1, \dots, u_p\}$ is an orthonormal basis for W, Then

$$\text{Proj}_W(y) = (y \cdot u_1)u_1 + (y \cdot u_2)u_2 + \dots + (y \cdot u_p)u_p$$

If $U = [u_1, \dots, u_p]$ then

$$\text{Proj}_W(y) = UU^T y \qquad (\forall y \in \mathbb{R}^n)$$

Pf: