§ 6.4 The Gram–Schmidt Process.

Example: Given

$$x_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Find an orthogonal basis for $W = \text{Span}(x_1, x_2)$.

Example: Let $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

Find an orthogonal basis for $W = \text{Span}(x_1, x_2, x_3)$

- <u>Theorem 11: (The Gram-Schmidt Process).</u>

Suppose $W \subseteq \mathbb{R}^n$ has basis $\{x_1, \dots, x_p\}$. Then,

$$v_1 = x_1$$
$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$
$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$
$$\vdots$$
$$v_p = x_p - \frac{x_p \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_p \cdot v_2}{v_2 \cdot v_2} v_2 - \cdots - \frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}} v_{p-1}$$

is an orthogonal basis for W.

Also,

$$\text{Span}(v_1, \dots, v_k) = \text{Span}(x_1, \dots, x_k)$$

for $1 \leq k \leq p$.

- <u>Pf:</u>

- **Note:**

If $\{v_1, \ldots, v_p\}$ is an orthogonal basis for $W \le \mathbb{R}^n$, then

$$\left\{ \frac{1}{\|v_1\|} v_1 , \ldots , \frac{1}{\|v_p\|} v_p \right\}$$

is an orthonormal basis for W.

- Pf:

- Theorem 12: (QR factorization).

Suppose A is $m \times n$ with L.I. columns.
Then $\exists$ Q, R s.t.

1. $A = QR$
2. Q is $m \times n$ and the columns of Q form an orthonormal basis for $\text{Col}(A)$.
3. R is $n \times n$ upper triangular.

§6.5 Least Squares Solution

Def: A least squares soln. of $A\vec{x}=\vec{b}$ is an $\hat{x} \in \mathbb{R}^n$ s.t.

$$\|b - A\hat{x}\| \le \|b - Ax\|$$

$\forall x \in \mathbb{R}^n$

Thm 13: The set of least squares solutions to $A\vec{x}=\vec{b}$ coincides w/ the non empty set of solutions of

$$A^T A \vec{x} = A^T b$$

Pf:

Take $\hat{b} = \mathrm{Proj}_{\mathrm{col}(A)}(\vec{b})$

Since $\hat{b} \in \mathrm{Col}(A)$ $\exists\ \hat{x}$ s.t.

$$A\hat{x} = \hat{b}$$

$$\boxed{A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}}$$

Note: $b-\hat{b} \perp \mathrm{Col}(A)$

$\Leftrightarrow$ $b - A\hat{x} \perp \vec{a}_i$ for $1 \le i \le n$

$\Leftrightarrow$ $\vec{a}_j^{\,t}(b-A\hat{x}) = 0 \quad \forall j$

$\Leftrightarrow$ $A^T(b-A\hat{x}) = 0$

$\Leftrightarrow$ $A^Tb - A^TA\hat{x} = 0 \Leftrightarrow A^TA\hat{x} = A^T\vec{b}$

Example: Find a least squares soln to $Ax = b$ where

$$A = \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

Soln.:

Note: The columns of A are orthogonal.

The orthog. proj. of b onto Col(A) is

$$\hat{b} = \text{Proj}_{\text{Col}(A)}(\vec{b}) = \frac{b \cdot a_1}{a_1 \cdot a_1}\vec{a}_1 + \frac{b \cdot a_2}{a_2 \cdot a_2}\vec{a}_2$$

$$=$$

Now we can solve $A\hat{x} = \hat{b}$ but we have done this above.

$$\hat{x} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

Thm 15: If A is $m \times n$ w/ LI col.'s. Let $A = QR$ be the QR-fact. of A. Then $\forall \vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ has the least squares soln.

$$\hat{x} = R^{-1} Q^T \vec{b}$$