MTHSC 412 Section 1.7 – Relations

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Definition

A *relation* on a set A is a subset of $A \times A$.

Notation

If R is a relation on A and $(a,b) \in R$ then we say that a is related to b by R and write aRb. If $(a,b) \notin R$ then we write $a \not R b$.

Example

Suppose that $A = \{a,b,c\}$ and $R = \{(a,b),(b,c),(c,a)\}$. Then we have aRb and $a \not R c$.

Example

Some well known relations on the integers are $<, >, \leq, \geq$ and $=$. Also, we have seen the $\subseteq$ relation on sets whose elements are sets.

Definition

A relation R on a nonempty set A is an *equivalence relation* if the following conditions hold for $x, y, z \in A$.

1. xRx for all $x \in A$. **(Reflexive Property)**
2. If xRy then yRx also. **(Symmetric Property)**
3. If xRy and yRz then xRz also. **(Transitive Property)**

Definition

We will consider the relation on $\mathbb{Z}$ defined as the set $\{(x,y) \in \mathbb{Z}^2 \quad | \quad (x-y) \text{ is divisible by } 4\}$. If (a,b) is in this set, we write $a \equiv b \pmod 4$.

Fact

$\equiv \pmod 4$ is an equivalence relation on $\mathbb{Z}$.

Proof.

Reflexive: Suppose that $x \in \mathbb{Z}$. Then $x - x = 0$ which is divisible by 4. So, $\equiv \pmod 4$ is reflexive.

Symmetric: Suppose that $x, y \in \mathbb{Z}$ and that $x \equiv y \pmod 4$. Then $(x - y) = 4k$ for some $k \in \mathbb{Z} \Rightarrow y - x = -4k = 4(-k)$. So, $y \equiv x \pmod 4$ as well and $\equiv \pmod 4$ is symmetric.

Transitive: Suppose that $x, y, z \in \mathbb{Z}$ with $x \equiv y \pmod 4$ and $y \equiv z \pmod 4$.

Then we have $x - y = 4k$ and $y - z = 4m$ form some $k, m \in \mathbb{Z}$.

So, $x - z = (x - y) + (y - z) = 4k + 4m = 4(k + m)$.

Thus $x \equiv z \pmod 4$ as well and $\equiv \pmod 4$ is transitive.

Since $\equiv \pmod 4$ is reflexive, symmetric and transitive, it is an equivalence relation. $\square$

Definition

Let R be an equivalence relation of a nonempty set A. For each $a \in A$ we define the *equivalence class* containing a as

$$[a] = \{x \in A \quad | \quad xRa\}.$$

Example

Let us consider again the relation $\equiv \pmod 4$ on $\mathbb{Z}$. Then,

$$\begin{aligned}
[0] &= \{\dots, -8, -4, 0, 4, 8, 12, \dots\} \\
[1] &= \{\dots, -7, -3, 1, 5, 9, 13, \dots\} \\
[2] &= \{\dots, -6, -2, 2, 6, 10, 14, \dots\} \\
[3] &= \{\dots, -5, -1, 3, 7, 11, 15, \dots\} \\
[4] &= [0], \quad [5] = [1]
\end{aligned}$$

Equivalence Classes Form a Partition

Definition

Let $\{A_i\}_{i\in I}$ be a collection of subsets of a nonempty set A. We say that $\{A_i\}_{i\in I}$ is a *partition* of A if the following conditions are satisfied.

1. $A_i \neq \emptyset$ for all $i \in I$.
2. $A = \cup_{i\in I} A_i$.
3. If $A_i \cap A_j \neq \emptyset$ then $A_i = A_j$.

Fact

- *If R is an equivalence relation on a nonempty set A then $\{[a] \quad | \quad a \in A\}$ is a partition of A.*
- *If $P = \{A_i\}_{i\in I}$ is a partition of A then there is an equivalence relation R on A such that the equivalence classes of R are precisely the parts A_i of P. To see this just define R by aRb if and only if a and b are in the same part of P.*