

MTHSC 412 SECTION 6.1 – IDEALS AND QUOTIENT RINGS

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DEFINITION

A subring I of a ring R is an ideal in R if for all $x \in I$ and $r \in R$, $xr, rx \in I$. If I is an ideal of R , we will write $I \trianglelefteq R$.

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NOTE

If R is any ring, then $\{0\}$ and R are ideals of R . These are referred to as *trivial ideals*.

EXAMPLE

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THEOREM

Suppose that $I \subseteq R$ where R is a ring. Then $I \trianglelefteq R$ if the following are true.

- 1 $I \neq \emptyset$.
- 2 $x, y \in I \Rightarrow (x + y) \in I$.
- 3 $x \in I \Rightarrow -x \in I$.
- 4 $x \in I; r \in R \Rightarrow xr, rx \in I$.

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Let $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z} \right\}$.

Show that $I \trianglelefteq S$.

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Suppose that R is a commutative ring with 1 and that $a \in R$. The *principal ideal generated by a* is defined by

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THEOREM

Suppose that R is a commutative ring with 1 and $a \in R$. Then $\langle a \rangle \trianglelefteq R$.

THEOREM ($\mathbb{Z}$ IS A PRINCIPAL IDEAL DOMAIN)

In the ring $\mathbb{Z}$, every ideal is principal.

DEFINITION

Suppose that R is a ring and $I \trianglelefteq R$. Then we consider the cosets

$$r + I = \{r + k \mid k \in I\}$$

and define addition and multiplication of these cosets as follows.

$$(a + I) + (b + I) = (a + b) + I; \quad (a + I)(b + I) = ab + I.$$

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THEOREM

Suppose that R is a ring and $I \trianglelefteq R$. Then $R/I = \{r + I \mid r \in R\}$ is a ring with addition and multiplication defined as above. R/I is called the quotient ring of R by I .

EXAMPLE

Consider $\mathbb{Z}/4\mathbb{Z}$.