952 Class Notes

Joel Clingempeel   Luke Giberson   Kevin James
Liem Nguyen   Sevasti Tagaris   Trevor Vilardi

Spring 2013

The Harmonic Series and Euler's Constant

9 January 2013

Definition. We define the *harmonic series*, H_n, as

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Lemma. *We have the following approximation:*

$$H_n - 1 \leq log(n) \leq H_n.$$

Proof. For reference, consider the following depiction of the function, $f(x) = \dfrac{1}{x}$, over the interval [0,7] :

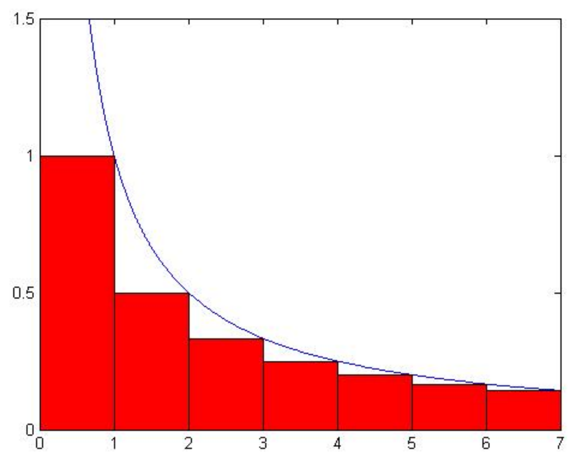


Figure 1: Right-hand Riemann approximation

Observe that H_n is given by the right-hand Riemann approximation of $\displaystyle\int_0^n \frac{dx}{x}$. In particular, subtracting off the area of the left-most rectangle, we have

$$\begin{aligned} H_n - 1 &\leq \int_1^n \frac{dx}{x} \\ &= log(n) - log(1) \\ &= log(n). \end{aligned}$$

For the other bound, consider the similar figure:

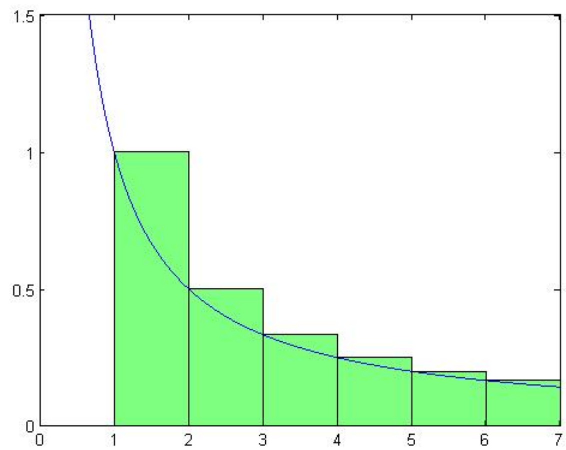


Figure 2: Left-hand Riemann approximation

In this case, H_n is given by the left-hand Riemann approximation of $\displaystyle\int_1^{n+1} \frac{dx}{x}$. So we have

$$\begin{aligned} H_n &\geq \int_1^{n+1} \frac{dx}{x} \\ &= log(n+1) - log(1) \\ &= log(n+1) \\ &> log(n). \end{aligned}$$

In this way, we conclude

$$H_n - 1 \leq log(n) \leq H_n.$$

$\square$

Of interest then, is to get our hands on this error term. To do so, we first recall a bit of notation.

Definition. Given two functions, $f(x)$ and $g(x)$, if there is a constant $c \in \mathbb{R}$ so that for some bound B, and for all $x \geq B$, we have that $|f(x)| \leq c \cdot g(x)$, then we write

$$f(x) = O(g(x)).$$

This is called *Big O notation.* We may also equivalently write $f(x) \ll g(x)$.

Theorem. *There is constant $\gamma \in \mathbb{R}$, which is known as Euler's constant, so that*

$$H_n = log(n) + \gamma + O\left(\frac{1}{n}\right).$$

Proof. Set $E_n = H_n - log(n+1)$, and refer to Figure 2. Geometrically speaking, E_n is the sum of the areas of the first n "triangular" regions above 1/x. First, note that as a sequence, E_n is strictly increasing, seeing as going further in the sequence results in gaining more area. Furthermore, for any n, we have $E_n \leq 1$ (one can imagine sliding each triangle left into the unit square formed by the axes). In this way, E_n is monotonic and bounded, so it has a limit. Set

$$\gamma = \lim_{n \to \infty} E_n.$$

Consider the quantity $\gamma - E_n$. We know that γ is the sum of the areas of all triangles, and E_n is the sum of the area of the first n triangles. Thus, $\gamma - E_n$ is the tail after the first n triangles. Using the same sliding argument as above, notice that

$$\gamma - E_n < \frac{1}{n+1} < \frac{1}{n}.$$

So we have

$$\begin{aligned} H_n &= log(n+1) + E_n \\ &= log(n+1) + \gamma + E_n - \gamma \\ &= log(n) + O\left(\frac{1}{n}\right) + \gamma + (E_n - \gamma) \\ &= log(n) + \gamma + O\left(\frac{1}{n}\right). \end{aligned}$$

The penultimate step will be shown with rigor in a later exercise, but arises from the fact that $log(n+1) \sim log(n)$. $\square$

Definition. We say that two functions f, g are *asymptotic* and write $f \sim g$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$

Example. As we will see, the Prime Number Theorem states that

$$\#\{p < x \mid \text{p is prime}\} \sim \frac{x}{log(x)}.$$

Example. We have

$$x^2 + x \sim \ x^2.$$

Remark. The last example makes clear that $\sim$ speaks only of relative error and nothing of absolute error.

Corollary. *We have the following immediate result:*

$$H_n \sim log(n) + \gamma.$$

Proof. Take $n \to \infty$ in the previous theorem, and the error term drops out. $\square$

Additive and Multiplicative Functions

11 January 2013

Notation: Little "o"

We say $f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

Exercises:

- **Suppose** $c \in \mathbb{R}$ **show** $\log(n + c) \sim \log(n)$**.**

  Pf: We evaluate the end behavior using l'Höpital's rule:

$$\lim_{n\to\infty} \frac{log(n+c)}{log(n)} = \lim_{n\to\infty} \frac{\frac{1}{n+c}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+c} = 1.$$

- $\log(x) = o(x^\epsilon)\ \forall \epsilon > 0$

  Pf: Let $\epsilon > 0$**.**
  We use l'Höpital's rule to evaluate the following:

$$\lim_{x\to\infty} \frac{\log x}{x^\epsilon} = \lim_{x\to\infty} \frac{1}{\epsilon x^\epsilon} = 0$$

  Hence $\log(x) = o(x^\epsilon)$.

Arithmetic Functions

Definition.

- A function $f : \mathbb{N} \to \mathbb{C}$ is said to be arithmetic.
- An arithmetic function f is additive if $f(mn) = f(m)+f(n)$ for $(m, n) = 1$. If this holds for all $m, n \in \mathbb{N}$ then f is completely additive.
- An arithmetic is multiplicative if $f(mn) = f(m)\dot{f}(n)$ for $(m, n) = 1$. If this holds for all $m, n \in \mathbb{N}$, then f is completely multiplicative.

Example.

- $\nu(n)$ = the number of distinct prime factors of n is an additive function. i.e. $\nu(12) = 2$.
- $\Omega(n)$ = the number of prime divisors of n counted with multiplicity is a completely additive function. ($\Omega(12) = 3$ because $12 = 2 \cdot 2 \cdot 3$).

Complex Analysis Notes

Recall: $e^{i\theta} = \cos\theta + i\sin\theta$.
So if $s = a + bi$ and $z \in \mathbb{R}_+$ then

$$z^s = z^{a+bi} = z^a(z^b)^i = z^a e^{ib\log z} = z^a(\cos(b\log z) + i\sin(b\log z)).$$

Definition.
$\sigma_s(n) = \sum_{d|n} d^s$ is multiplicative but not completely multiplicative.

Proof. First the counterexample: $\sigma_s(4) = 1^s + 2^s + 4^x \neq (1^s + 2^s)^2 = \sigma_s(2)\sigma_s(2)$. Now suppose $(m, n) = 1$. Then

$$\sigma_s(mn) = \sum_{d|mn} d^s = \sum_{d_1|m}\sum_{d_2|n}(d_1d_2)^s$$

Exercise: Justify this, i.e $\{d \mid mn\} \leftrightarrow \{(d_1, d_2) \mid d_1 \mid m, d_2 \mid n\}$.

Proof. Since $(m, n) = 1$, each d uniquely determines prime factors that comprise d_1, d_2 so that $d = d_1d_2$ and $d_1 \mid m$ while $d_2 \mid n$. Thus, we have

$$\sum_{d|mn} d^s = \sum_{d_1d_2|mn}(d_1d_2)^s = \sum_{d_1|m}\sum_{d_2|n}(d_1d_2)^s.$$

$\square$

$$\sigma_s(mn) = \left(\sum_{d_1|m} d_1^s\right)\left(\sum_{d_2|n} d_2^s\right) = \sigma_s(m)\sigma_s(n)$$

Note: If $n = p_1^{a_1}...p_k^{a_k}$ then apply the definition of multiplicative function and induction we get

$$\begin{aligned}
\sigma_s(n) = \prod_{i=1}^{k} \sigma_s(p_i^{a_i}) &= \prod_{i=1}^{k} (1^s + {p_i}^s + {p_i}^{2s} + ... + {p_i}^{sa_i}) \\
&= \prod_{i=1}^{k} \left(\sum_{j=0}^{a_i} (p_i^s)^j\right) \\
&= \prod_{i=1}^{k} \frac{(p_i^s)^{a_i+1} - 1}{p_i^s - 1} \text{ (provided } s \neq 0).
\end{aligned}$$

If $s = 0$, then $\sigma_0(n) = \prod_{i=1}^{k} \sigma_0(p_i^{a_i}) = \prod_{i=1}^{k} (a_i + 1)$.

$\square$

Definition.

- $\mu(n) = \begin{cases} (-1)^{\nu(n)} & \text{if } n \text{ is square free} \\ 0 & \text{otherwise} \end{cases}$
- $\phi(n) = \#\{1 \leq m \leq n \mid (m,n) = 1\} = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$.
- Von Mangoldt function

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^\alpha \text{ for some } \alpha \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

e.g $\Lambda(12) = 0$.

$$\Lambda(8) = \log(2)$$

$$\Lambda(9) = \log(3)$$

$$\Lambda(72) = 0$$

Lemma.

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \textit{if } n = 1 \\ 0 & \textit{otherwise} \end{cases}$$

Proof.

Suppose $u = p_1^{a_1} ... p_k^{a_k} > 1$ Then

$$\begin{aligned}
\sum_{d|n} \mu(d) = \sum_{i_1=0}^{a_1} \sum_{i_2=0}^{a_2} ... \sum_{i_k=0}^{a_k} \mu(p_1^{i_1} ... p_k^{i_k}) &= \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} ... \sum_{i_k=0}^{1} \mu(p_1^{i_1} ... p_k^{i_k}) \\
&= \prod_{j=1}^{k} (\mu(1) + \mu(p_j)) \\
&= \prod_{j=1}^{k} (1 - 1) \\
&= 0.
\end{aligned}$$

If $n = 1$, then $\sum_{d|1} \mu(d) = \mu(1) = 1$

□

Theorem. *(Möbius Inversion)*

$$f(n) = \sum_{d|n} g(d) \Leftrightarrow g(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right)$$

Proof.

($\Rightarrow$): Suppose $f(n) = \sum_{d|n} g(d)$.

Then,

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d) \sum_{e|\frac{n}{d}} g(e)$$

$$= \sum_{n=deh} \mu(d) g(e) = \sum_{e|n} g(e) \left(\sum_{d|\frac{n}{e}} \mu(d) \right) = g(n)$$

($\Leftarrow$): Suppose $g(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right)$

Then

$$\begin{aligned}
\sum_{d|n} g(d) = \sum_{d|n} \sum_{e|d} \mu(e) f\left(\frac{d}{e}\right) &= \sum_{ehk=n} \mu(e) f(h) \\
&= \sum_{h|n} f(h) \sum_{j|\frac{n}{n}} (\mu(j)) \\
&= f(n)
\end{aligned}$$

$\square$

Remark. The following are just rearrangements:

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d) = \sum_{n=ed} \mu(e) f(d).$$

Möbius Inversion

14 January 2013

Exercise: $\sum_{d|n} \phi(d) = n$

Proof. Recall from group theory the group $\mathbb{Z}_n$. A preliminary result is that if some $d \mid n$, then there are exactly $\phi(d)$ elements with order d in $\mathbb{Z}_n$. Since the order of an element must divide the order of a group, summing $\phi(d)$ over all divisors d of n counts the elements in $\mathbb{Z}_n$, of which there are n of them. □

Lemma.

$$\frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}$$

Proof. Equivalently, we will show $\phi(n) = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right)$.

Take $f(m) = m$ and $g(m) = \phi(m)$.
Then, using the previous fact, we have $n = f(n) = \sum_{d|n} g(d)$.

Now, using Möbius Inversion, we have:

$$\begin{aligned}
\phi(m) &= g(m) \\
&= \sum_{d|m} \mu(d) f\left(\frac{m}{d}\right) \\
&= \sum_{d|m} \mu(d) \left(\frac{m}{d}\right).
\end{aligned}$$

□

Remark. Suppose $n = p_1^{a_1} \cdots p_k^{a_k}$ and let f be multiplicative.

1. $f(n) = \prod_{i=1}^{k} f(p_i^{a_i})$.
2. $g(n) = \sum_{d|n} f(d) = \sum_{i_1=0}^{a_1} \sum_{i_2=0}^{a_2} ... \sum_{i_k=0}^{a_k} f(p_1^{i_1}) \cdots f(p_k^{i_k})$.
   We do this because the divisors look like $p_i^{a_i}$. Then, rearrange the terms to get a k-fold product of the sums so that $g(n) = \prod_{l=1}^{k} \left(\sum_{i_l=0}^{a_l} f(p_l^{i_l}) \right)$.

3. If f is multiplicative, $g(n) = \sum_{d|n} f(d)$ is multiplicative.

   Let $m = q_1^{b_1} \dots q_l^{b_l}$, and suppose $(m, n) = 1$. Then

$$\begin{aligned} g(nm) &= \sum_{d|nm} f(d) \\ &= \sum_{i_1=0}^{a_1} \dots \sum_{i_k=0}^{a_k} \sum_{j_1=0}^{b_1} \dots \sum_{j_l=0}^{b_l} f(p_1^{i_1}) \dots f(p_k^{j_k}) f(q_1^{b_1}) \dots f(q_l^{j_l}) \\ &= \prod_{i=1}^{k} \left(\sum_{t=0}^{a_i} f(p_i^{t_i}) \right) \cdot \prod_{j=1}^{l} \left(\sum_{s=0}^{b_j} f(q_j^{s_j}) \right) \\ &= g(n)g(m). \end{aligned}$$

   This means that in Möbius Inversion, either both f and g are multiplicative, or neither are.

Exercise:
Recall:

$$\Lambda(m) = \begin{cases} \log(p) & \text{if } m = p^\alpha \\ 0 & \text{otherwise} \end{cases}$$

1. Show that $\sum_{d|n} \Lambda(d) = \log(n)$.

   Proof. Let $n = p_1^{a_1} \dots p_k^{a_k}$. Observe that

$$\begin{aligned} \sum_{d|n} \Lambda(d) &= \sum_{i_1=0}^{a_1} \dots \sum_{i_k=0}^{a_k} \Lambda(p_1^{i_1} \dots p_k^{i_k}) \\ &= \left[\sum_{i_1=0}^{a_1} \Lambda(p_1^{i_1}) \right] \dots \left[\sum_{i_k=0}^{a_k} \Lambda(p_k^{i_k}) \right] \\ &= \sum_{i_1=1}^{a_1} log(p_1) \dots \sum_{i_k=1}^{a_k} log(p_k) \\ &= a_1 log(p_1) \dots a_k log(p_k) \\ &= log(n). \end{aligned}$$

   The second step is the key, and though it looks like it relies on the property of multiplicity (with $\Lambda(n)$ doesn't have), I believe it holds because for distinct primes p, q, we have $\Lambda(pq) = 0$, which kills many of those sums. Thoughts? $\square$

2. Using Möbius Inversion, deduce that $\Lambda(n) = -\sum_{d|n} \mu(d) \log(d)$.

Proof. Set $f(m) = log(m)$ and $g(m) = \Lambda(m)$. Then using the previous exercise and Möbius Inversion, we have

$$\begin{aligned} g(n) = \Lambda(n) &= \sum_{d|n} \mu(d) log\left(\frac{n}{d}\right) \\ &= \sum_{d|n} \mu(d) log(n) - \sum_{d|n} \mu(d) log(d) \\ &= log(n) \sum_{d|n} \mu(d) - \sum_{d|n} \mu(d) log(d) \\ &= 0 - \sum_{d|n} \mu(d) log(d) \\ &= -\sum_{d|n} \mu(d) log(d). \end{aligned}$$

□

Exercise:
Show that for positive x,

$$G(x) = \sum_{n \leq x} F\left(\frac{x}{n}\right) \iff F(x) = \sum_{n \leq x} \mu(n)\ G\left(\frac{x}{n}\right).$$

Fact.

1. Suppose $\sum_{k=1}^{\infty} d_3(k)|f(kx)| < \infty$ and $g(x) = \sum_{m=1}^{\infty} f(mx)$.

   Then, $f(x) = \sum_{n=1}^{\infty} \mu(n) g(nx)$.

2. Suppose $\sum_{k=1}^{\infty} d_3(k)|g(kx)| < \infty$ and $f(x) = \sum_{n=1}^{\infty} \mu(n) g(nx)$.

   Then, $g(x) = \sum_{m=1}^{\infty} f(mx)$.

Proof. $\Rightarrow$ Suppose $g(x) = \sum_{m=1}^{\infty} f(mx)$. Then,

$$\begin{aligned}\sum_{n=1}^{\infty} \mu(n) g(nx) &= \sum_{n=1}^{\infty} \mu(n) \sum_{m=1}^{\infty} f(mnx) \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mu(n) f(mnx).\end{aligned}$$

Let $mn = r$ so that $n \mid r$. Then, rearrange* to get $\sum_{r=1}^{\infty} f(rx) \sum_{n|r} \mu(n)$.

Recall:

$$\sum_{n|r} \mu(n) = \begin{cases} 1 & r = 1 \\ 0 & \text{otherwise} \end{cases}$$

All that remains is when $r = 1$ so that $\sum_{r=1}^{\infty} f(rx) \sum_{n|r} \mu(n) = f(x)$.

*Rearrangement is justified because absolute convergence of the sum:
We have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |f(mnx)| \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\mu(n) f(mnx)|$$

so we can see: $\sum_{r \leq 1}^{\infty} |f(rx)| d(r) \leq \sum_{r=1}^{\infty} d_3(r) |f(rx)| < \infty$ by assumption. $\square$

Exercise 1.11 Liouville's Function $\lambda(n) = (-1)^{\Omega(n)}$. Show that

$$\sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let $n = p_1^{a_1} \dots p_k^{a_k}$. Then we have

$$\begin{aligned}\sum_{d|n} \lambda(d) &= \sum_{i_1=0}^{a_1} \dots \sum_{i_k=0}^{a_k} \lambda(p_1^{i_1} \dots p_k^{i_k}) \\ &= \sum_{i_1=0}^{a_1} \dots \sum_{i_k=0}^{a_k} (-1)^{\sum_{j=1}^{k} i_j} \\ &= \prod_{j=1}^{k} \left(\sum_{i_j=0}^{a_j} (-1)^{i_j} \right).\end{aligned}$$

Notice that the inside sum is 1 when a_j is even. So for the entire product to be 1, we require each of $a_1, \dots, a_k$ to be even – which is equivalent to requiring that n be a square. The product vanishes otherwise. $\square$

Exercise 1.12 Ramanujan Sums

Define $C_n(m) = \sum_{\substack{1 \le h \le n \\ (h,n)=1}} e\left(\frac{hm}{n}\right)$ and $e(t) = e^{2\pi i t}$.

We want to find a better way to express this.

Put

$$\begin{aligned} g(m) &= \sum_{1 \le h \le n} e\left(\frac{hm}{n}\right) \\ &= e\left(\frac{m}{n}\right)\left(\frac{e\left(\frac{m}{n}\right)^n - 1}{e\left(\frac{m}{n}\right) - 1}\right) \\ &= 0 \end{aligned}$$

If $n \not| \, m$, this is our result.

If $n \mid m$, then we get $\sum_{1 \le h \le n} 1$ because we are considering $e^{\pi i}$ raised to an integer which is 1 so we are consdering $\sum_{1 \le h \le n} 1 = n$. This gives us a nicer formula so that

$$g_m(n) = \begin{cases} n & \text{if } n \mid m \\ 0 & \text{otherwise} \end{cases}$$

Note: $g_m(n) = \sum_{d|n} \sum_{\substack{1 \le h \le n \\ (h,n)=d}} e\left(\frac{hm}{n}\right)$ so let $h'd = h$ and $n'd = n$ so we are now considering $\sum_{d|n} \sum_{\substack{1 \le h' \le n' \\ (h',n')=1}} e\left(\frac{h'dm}{dn'}\right) = \sum_{d|n} C_{n'}(m)$.

So, we have $g_m(n) = \sum_{d|n} C_{\frac{n}{d}}(m)$ where $\frac{n}{d}$ is the variable.

Using Möbius Inversion, we can get our hands on the right hand side so that

$$\begin{aligned} C_n(m) &= \sum_{d|n} \mu(d) g_m\left(\frac{n}{d}\right) \\ &= \sum_{d|n} \mu\left(\frac{n}{d}\right) g_m(d) \\ &= \sum_{d|(m,n)} \mu\left(\frac{n}{d}\right)(d). \end{aligned}$$

Note: Everything is killed but $d \mid m$ because if $d \not| \, m$, $g_m(d) = 0$.

Formal Dirichlet Series

01/16/2013

Ramanujan Sums Continued

Recall

$$C_n(m) = \sum_{\substack{1 \le h \le n \\ (h,n)=1}} e\left(\frac{hm}{n}\right) = \sum_{d|(n,m)} \mu\left(\frac{n}{d}\right) d$$

For p prime we have

$$C_p(m) = \sum_{d|(p,m)} \mu\left(\frac{p}{d}\right) = \begin{cases} -1 & p \nmid m \\ p-1 & p|m \end{cases}$$

and

$$C_n(1) = \sum_{\substack{1 \le h \le n \\ (h,n)=1}} e\left(\frac{h}{n}\right) = \mu(n)$$

Exercise 1.14: Let $\delta = (n,m)$. Show

$$C_n(m) = \frac{\mu\left(\frac{n}{\delta}\right)\varphi(n)}{\varphi\left(\frac{n}{\delta}\right)}$$

Dirichlet Series

Definition. If f is an arithmetic function, then its formal Dirichlet series is

$$D(f,s) = \sum_{n=1}^{\infty} f(n)n^{-s} = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

Fact.

1. $D(f,s) + D(g,s) = D(f+g,s)$
2. $D(f,s)D(g,s) = D(f*g,s)$
   where $f*g(n) = \sum_{de=n} f(d)g(e)$ is called the convolution of f and g.

Definition.

$$\delta(n) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{otherwise} \end{cases}$$

Note that $f*\delta = f$. So δ is an identity for the operation of convolution.

Exercise

1. If f and g are multiplicative, is $f*g$ multiplicative?

Proof. Suppose $(m,n)=1$. In an exercise from the second lecture, recall that given relatively prime m,n, we discussed the one-to-one correspondence between divisors of mn and ordered pairs of divisors (d_1,d_2), where $d_1 \mid m$ and $d_2 \mid n$. This same idea justifies the key summation manipulation below:

$$\begin{aligned}(f\star g)(m)\cdot(f\star g)(n) &= \sum_{ed=m} f(e)g(d)\cdot\sum_{bc=n} f(c)g(b)\\ &= \sum_{\substack{ed=m\\ bc=n}} f(e)g(d)f(c)g(b)\\ &= \sum_{\substack{ed=m\\ bc=n}} f(ec)g(db)\\ &= \sum_{st=mn} f(s)g(t)\\ &= (f\star g)(mn).\end{aligned}$$

□

2. Is $*$ associative?

3. Are multiplicative functions invertible under $*$ (i.e. can we solve $f*g=\delta$ for g given f)?

Fact

If f is multiplicative, then

$$D(f,s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\prod_{p,\ p\text{ prime}}\left(\sum_{k=0}^{\infty}\frac{f(p^k)}{p^{ks}}\right)=\prod_{p,\ p\text{ prime}}(1+f(p)p^{-s}+f(p^2)p^{-2s}+\ldots)$$

Definition.

$$D(1,s)=\sum_{n\geq 1}\frac{1}{n^s}=\zeta(s)$$

Note that

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}=\prod_{p,\ p\text{ prime}}\left(\sum_{k=0}^{\infty}\frac{1}{p^{ks}}\right)=\prod_{p,\ p\text{ prime}}\left(\frac{1}{1-\frac{1}{p^s}}\right)=\prod_{p,\ p\text{ prime}}(1-p^{-s})^{-1}$$

where the third equality comes from the fact that the inner sum is a geometric series in p^s.

Fact

$$D(\mu,s)=\frac{1}{\zeta(s)}$$

Proof.

$$D(\mu, s) = \prod_{p,\ p \text{ prime}} \left(\sum_{k=0}^{\infty} \frac{\mu(p^k)}{p^{ks}} \right) = \prod_{p,\ p \text{ prime}} (1 - p^{-s}) = \frac{1}{\zeta(s)}$$

□

Now note that

$$\lim_{s \to 1^+} \zeta(s) = \lim_{x \to \infty} \sum_{n \leq x} \frac{1}{n} \geq \lim_{x \to \infty} \log(x) = \infty.$$

This implies that $\prod_p \left(\frac{1}{1 - \frac{1}{p^s}} \right) \to \infty$ as $s \to 1^+$. If we let $s = 1$ we get

$$\frac{1}{1 - \frac{1}{p^s}} = \frac{1}{1 - \frac{1}{p}} = \frac{p}{p-1}.$$

But we saw above that if we take the product of $\frac{p}{p-1}$ over all primes it will go to infinity. That can only happen if we have an infinite number of terms in the product. Therefore there are infinitely many primes.

Fact 1.2.3:

$$D(\Lambda, s) = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} = \frac{-\zeta'(s)}{\zeta(s)} = \sum_{\substack{p^k \geq 1 \\ p, \text{ prime}}} \frac{\log(p)}{p^{ks}}$$

Proof. Differentiating $\zeta(s)$ yields

$$\zeta'(s) = -\sum_{n \geq 1} \frac{\log(n)}{n^s}.$$

Thus $-\zeta'(s) = D(\log, s)$. Also, as we saw before $\frac{1}{\zeta(s)} = D(\mu, s)$. So

$$\frac{-\zeta'(s)}{\zeta(s)} = D(\log, s) D(\mu, s) = D(\log * \mu, s) = D(\Lambda, s)$$

□

Exercise Prove the above blue equality.

Proof. In an earlier exercise, we showed that

$$\Lambda(n) = \sum_{d|n} \mu(d) \log\left(\frac{n}{d}\right),$$

which is precisely our definition of convolution. □

Orders of Some Arithmetic Functions

18 January 2013

Fact Suppose $f(n) = \sum_{d|n} g(n) = g \star 1$. Then we have

$$D(f, s) = D(g \star 1, s) = D(g, s)D(1, s) = D(g, s)\zeta(s).$$

Exercise Show that for $\lambda(n) = (-1)^{\Omega(n)}$, we have

$$D(\lambda, s) = \frac{\zeta(2s)}{\zeta(s)}.$$

Proof. Keeping in mind that Liouville's function is multiplicative, we compute

$$\begin{aligned} D(\lambda, s) &= \sum_{n \geq 1} \frac{\lambda(n)}{n^s} \\ &= \prod_p \left(\sum_{k=0}^{\infty} \frac{\lambda(p^k)}{p^{ks}} \right) \\ &= \prod_p \left(\sum_{k=0}^{\infty} \left(\frac{-1}{p^s} \right)^k \right) \\ &= \prod_p \left(\frac{1}{1 + p^{-s}} \right) \\ &= \prod_p \left(\frac{1 + p^{-s}}{1 - p^{-2s}} \right) \\ &= \frac{\zeta(2s)}{\zeta(s)}. \end{aligned}$$

$\square$

Exercise We have the following identity:

$$D(2^\nu, s) = \sum_{n \geq 1} \frac{2^{\nu(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)}.$$

Proof. First, note that since $\nu(n)$ is additive, the function $2^{\nu(n)}$ is multiplicative.

We have

$$\begin{aligned}
\sum_{n\geq 1}\frac{2^{\nu(n)}}{n^s} &= \prod_p\left(\sum_{k=0}^{\infty}\frac{2^{\nu(p^k)}}{p^{ks}}\right)\\
&= \prod_p\left(1+\sum_{k=1}^{\infty}2\left(\frac{1}{p^s}\right)^k\right)\\
&= \prod_p\left(1+\frac{2p^{-s}}{1-p^{-s}}\right)\\
&= \prod_p\left(\frac{1+p^{-s}}{1-p^{-s}}\right)\\
&= \prod_p\left(\frac{1-p^{-2s}}{(1-p^{-s})^2}\right)\\
&= \frac{\zeta^2(s)}{\zeta(2s)}.
\end{aligned}$$

□

Fact The following computation gives the Dirichlet series for $|\mu(n)|$. Observe that

$$\begin{aligned}
D(|\mu|,s) &= \sum_{n\geq 1}\frac{|\mu(n)|}{n^s}\\
&= \prod_p\sum_{k=0}^{\infty}\frac{|\mu(p^k)|}{p^{ks}}\\
&= \prod_p\left(1+\frac{1}{p^s}\right)\\
&= \prod_p\left(\frac{1-\frac{1}{p^{2s}}}{1-\frac{1}{p^s}}\right)\\
&= \frac{\zeta(s)}{\zeta(2s)}.
\end{aligned}$$

Ramanujan's Identity

$$\sum_{n=0}^{\infty}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\frac{\gamma^{n+1}-\delta^{n+1}}{\gamma-\delta}T^n = \frac{1-\alpha\beta\gamma\delta T^2}{(1-\alpha\gamma T)(1-\alpha\delta T)(1-\beta\gamma T)(1-\beta\delta T)}$$

Fact 1.2.8

$$\sum_{n=1}^{\infty}\frac{d^2(n)}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)}$$

Proof. Write

$$\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}=\alpha^n+\alpha^{n-1}\beta+...+\alpha\beta^{n-1}+\beta^n.$$

Take $\alpha=\beta=\gamma=\delta=1$ in Ramanujan's identity to get

$$\sum_{n=0}^{\infty}(n+1)^2T^n=\frac{1-T^2}{(1-T)^4}.$$

Then

$$\sum_{n=1}^{\infty}\frac{d^2(n)}{n^s}=\prod_p\sum_{k=0}^{\infty}\frac{d^2(p^k)}{p^{ks}}=\prod_p\sum_{k=0}^{\infty}\frac{(k+1)^2}{p^{ks}}.$$

Taking $T=\dfrac{1}{p^s}$ yields

$$\prod_p\frac{1-\frac{1}{p^{2s}}}{(1-\frac{1}{p^s})^4}=\frac{\zeta^4(s)}{\zeta(2s)}.$$

□

Exercise 1.2.9

Using Ramanujan's identity, show for $a,b\in\mathbb{C}$ that

$$\sum_{n=1}^{\infty}\sigma_a(n)\sigma_b(n)\frac{1}{n^s}=\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}.$$

Proof. If we write the right-hand side as an Euler product, then the factor for a fixed prime p is given by

$$\frac{1-p^{-2s}p^ap^b}{(1-p^{-s})(1-p^{-s}p^a)(1-p^{-s}p^b)(1-p^{-s}p^ap^b)}.$$

Now applying Ramanujan's identity with $\alpha=1$, $\beta=p^b$, $\gamma=1$, $\delta=p^a$, and $T=p^{-s}$ reduces this to

$$\sum_{m=0}^{\infty}\frac{1-p^{a(m+1)}}{1-p^a}\frac{1-p^{b(m+1)}}{1-p^b}p^{-ms}=\sum_{m=0}^{\infty}(1+p^a+p^{2a}+...+p^{am})(1+p^b+p^{2b}+...+p^{bm})p^{-ms}.$$

If we set

$$A_{p,m}=(1+p^a+p^{2a}+...+p^{am})(1+p^b+p^{2b}+...+p^{bm})p^{-ms},$$

then

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} = \prod_p \sum_{m=0}^{\infty} A_{p,m} = \sum_{n=0}^{\infty} \prod_{p^m || n} A_{p,m} =$$

$$\sum_{n=1}^{\infty} (\prod_{p^m||n} (1+p^a+p^{2a}+...+p^{am}))(\prod_{p^m||n} (1+p^b+p^{2b}+...+p^{bm})) \prod_{p^m||n} p^{-ms} = \sum_{n=1}^{\infty} \sigma_a(n)\sigma_b(n)\frac{1}{n^s}.$$

□

Exercise

Let

$$q_k(n) = \begin{cases} 1 & n = k^{th} \text{ power} \\ 0 & \text{otherwise} \end{cases}.$$

Then $D(q_k, s) = \dfrac{\zeta(s)}{\zeta(ks)}$.

Proof. It's important to note that $q_k(n)$ is multiplicative. So we have

$$\sum_{n=1}^{\infty} q_k(n)n^{-s} = \prod_p (1 + p^{-s} + p^{-2s} + ...p^{-(k-1)s}) = \prod_p \frac{1 - p^{-ks}}{1 - p^{-s}} = \frac{\zeta(s)}{\zeta(ks)}.$$

□

Fact 1.3.1

$$d(n) = \sum_{d|n} 1 = \sum_{d|n, d \leq \sqrt{n}} 1 \leq 2\sqrt{n}$$

Fact 1.3.2 For every $\epsilon > 0$, there exists a constant $c(\epsilon) \in \mathbb{R}$ such that $d(n) \leq c(\epsilon)n^{\epsilon}$.

Proof.

$$\frac{d(n)}{n^{\epsilon}} = \prod_{p^{\alpha}|n} \frac{d(p^{\alpha})}{p^{\alpha\epsilon}} = \prod_{p^{\alpha}|n} \frac{\alpha + 1}{p^{\alpha\epsilon}}$$

If $p > 2^{\frac{1}{\epsilon}}$, then $p^{\epsilon} > 2$ so $p^{\alpha\epsilon} > 2^{\alpha}$. Therefore $\dfrac{1}{p^{\alpha\epsilon}} < \dfrac{1}{2^{\alpha}}$ so $\dfrac{\alpha+1}{p^{\alpha\epsilon}} < \dfrac{\alpha+1}{2^{\alpha}} \leq 1$ and then $\displaystyle\prod_{p > 2^{\frac{1}{\epsilon}}} \frac{\alpha+1}{p^{\alpha\epsilon}} < 1$. For $p \leq 2^{\frac{1}{\epsilon}}$,

$$\frac{\alpha+1}{p^{\alpha\epsilon}} = \frac{\alpha}{p^{\alpha\epsilon}} + \frac{1}{p^{\alpha\epsilon}} \leq 1 + \frac{\alpha}{p^{\alpha\epsilon}} \leq 1 + \frac{\alpha}{2^{\alpha\epsilon}} = 1 + \frac{\alpha}{e^{\log(2)\alpha\epsilon}}.$$

Now for any $y \in \mathbb{R}$, one has $e^y > y$ since $e^y = 1 + y +$ higher order terms. Hence

$$1 + \frac{\alpha}{e^{\log(2)\alpha\epsilon}} \leq 1 + \frac{\alpha}{\log(2)\alpha\epsilon} = 1 + \frac{1}{\epsilon\log(2)}.$$

Thus

$$\prod_{p^\alpha || n, p \leq 2^{\frac{1}{\epsilon}}} \frac{\alpha+1}{p^{\alpha\epsilon}} < \prod_{p^\alpha || n, p \leq 2^{\frac{1}{\epsilon}}} (1 + \frac{1}{\epsilon\log(2)}) < \prod_{p \leq 2^{\frac{1}{\epsilon}}} (1 + \frac{1}{\epsilon\log(2)}) = (1 + \frac{1}{\epsilon\log(2)})^{\pi(2^{\frac{1}{\epsilon}})}.$$

Now take

$$c(\epsilon) = \prod_{p \leq 2^{\frac{1}{\epsilon}}} (1 + \frac{1}{\epsilon\log(2)}).$$

Then

$$\frac{d(n)}{n^\epsilon} = \prod_{p^\alpha || n} \frac{\alpha+1}{p^{\alpha\epsilon}} < c(\epsilon) \prod_{p^\alpha || n, p > 2^{\frac{1}{\epsilon}}} \frac{\alpha+1}{p^{\alpha\epsilon}} < c(\epsilon).$$

□

Exercise 1.3.3

For any $n > 0$, show that

$$d(n) < 2^{\frac{(1+\epsilon)\log(n)}{\log(\log(n))}}.$$

Hint: Take

$$\epsilon = \frac{(1 + \frac{n}{2})\log(2)}{\log(\log(n))}.$$

Note: We showed

$$d(n) < c(\epsilon)n^\epsilon = 2^{\frac{\log(c(\epsilon))}{\log(2)} + \frac{2\log(n)}{\log(2)}}.$$

Further Bounds on Arithmetic Functions

23 January 2013

We continue with our work in bounding specific arithmetic functions.

Fact. *We have the following bound on $\sigma_1(n)$:*

$$\sigma_1(n) \leq n(log(n) + 1).$$

Proof. Observe that

$$\begin{aligned}
\sigma_1(n) &= \sum_{d|n} d \\
&= \sum_{d|n} \frac{n}{d} \\
&= n \sum_{d|n} \frac{1}{d} \\
&\leq n \sum_{d=1}^{n} \frac{1}{d} \\
&\leq n \left(log(n) + \gamma + O\left(\frac{1}{n}\right)\right),
\end{aligned}$$

where this bound on the harmonic series comes from the first lecture. Recall that the coefficient on the error term is a 1, and so using the Euler constant approximation $\gamma \approx 0.5772...$, we may conclude that for $n \geq 3$

$$\sigma_1(n) \leq n(log(n) + 1).$$

The cases for $n = 1$ and $n = 2$ are easily verified. □

Remark. A very simple lower bound is $\sigma_1(n) = n \sum_{d|n} \frac{1}{d} \geq n$. Therefore,

$$n \leq \sigma_1(n) \leq n(log(n) + 1).$$

Switching our emphasis to $\phi(n)$, we have the immediate upper bound $\phi(n) \leq n$, where equality is reached only on $\phi(1) = 1$. In regards to how small $\phi(n)$ gets, we present no concrete result. Observe, however, that

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) = n \prod_{p|n} \left(\frac{p-1}{p}\right),$$

and so $\phi(n)$ is small when many distinct primes divide n.

We do have a nice relationship between $\sigma_1(n)$ and $\phi(n)$:

Fact. *There are constants $c_1, c_2 \in \mathbb{R}$ so that*

$$c_1 n^2 \leq \phi(n)\sigma_1(n) \leq c_2 n^2.$$

Proof. We begin by expressing these arithmetic functions in a more accessible form. We know that

$$\sigma_1(n) = \prod_{p^\alpha || n} (1 + p + ... + p^\alpha).$$

Therefore, dividing both sides of this identity by n yields

$$
\begin{aligned}
\frac{\sigma_1(n)}{n} &= \prod_{p^\alpha || n} \left(\frac{1}{p^\alpha} + \frac{1}{p^{\alpha-1}} + ... + 1 \right) \\
&= \prod_{p^\alpha || n} \left(\sum_{k=0}^{\alpha} \frac{1}{p^k} \right) \\
&= \prod_{p^\alpha || n} \left(\frac{\frac{1}{p^{\alpha+1}} - 1}{\frac{1}{p} - 1} \right) \\
&= \prod_{p^\alpha || n} \left(\frac{p^{\alpha+1} - 1}{p^\alpha (1-p)} \right).
\end{aligned}
$$

For $\phi(n)$, we have

$$
\frac{\phi(n)}{n} = \prod_{p|n} \left(\frac{p-1}{p} \right) = \prod_{p^\alpha || n} \left(\frac{p-1}{p} \right).
$$

Thus, multiplying these expressions together gives

$$
\begin{aligned}
\frac{\phi(n)\sigma_1(n)}{n^2} &= \prod_{p^\alpha || n} \left(\frac{p-1}{p} \right) \left(\frac{p^{\alpha+1} - 1}{p^\alpha (1-p)} \right) \\
&= \prod_{p^\alpha || n} \left(\frac{p^{\alpha+1} - 1}{p^{\alpha+1}} \right).
\end{aligned}
$$

We immediately notice that each factor in this product is less than 1, so the product as a whole is less than one. Therefore we have

$$
\phi(n)\sigma_1(n) \leq n^2.
$$

Hence, in the context of the problem statement, $c_2 = 1$.

For the lower bound, we'll need to do a bit more work with our product. First, we'll rewrite it as

$$
\frac{\phi(n)\sigma_1(n)}{n^2} = \prod_{p^\alpha || n} \left(\frac{p^{\alpha+1} - 1}{p^{\alpha+1}} \right) = \prod_{p^\alpha || n} \left(1 - \frac{1}{p^{\alpha+1}} \right),
$$

so that multiplying by n^2 yields the following approximation:

$$\begin{aligned}
\phi(n)\sigma_1(n) &= n^2 \prod_{p^\alpha || n} \left(1 - \frac{1}{p^{\alpha+1}}\right) \\
&\geq \prod_{p^\alpha || n} \left(1 - \frac{1}{p^{\alpha+1}}\right) \\
&\geq \prod_{p^\alpha || n} \left(1 - \frac{1}{p^2}\right) \\
&\geq \prod_{p} \left(1 - \frac{1}{p^2}\right) \\
&= \frac{1}{\zeta(2)}.
\end{aligned}$$

Here, we note that Euler first computed

$$\frac{1}{\zeta(2)} = \sum_{n \geq 1} \frac{1}{n^2} = \frac{6}{\pi},$$

and so we may set this as c_2, which completes the proof. $\square$

There is a tiny bit of hand-waving going in the final steps of the previous proof. Without citing Euler, how might we have known that the infinite product above actually converged?

Definition. Suppose z_n is non-zero for all $n \in \mathbb{N}$. Set $P_k = \prod_{n=1}^{k} z_n$ as the kth partial product of z_n. We define the *infinite product of* z_n by

$$z = \prod_{n=1}^{\infty} z_n = \lim_{k \to \infty} \left(\prod_{n=1}^{k} z_n \right).$$

Notice that for any k, we have

$$z_k = \frac{P_k}{P_{k-1}}.$$

This means that as $k \to \infty$, we require

$$z_k = \frac{z}{z} = 1.$$

In other words, the sequence must tend to 1 in order for the product of the terms of the sequence to converge.

25 January 2013

$$z = \prod_n z_n$$

$$p_k = \prod_{n=1}^{k} z_n$$

$$\frac{p_k}{p_{k-1}} = z_k$$

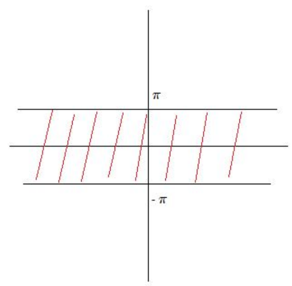
Since $p_k \to z$, $z_k \to 1$ as $k \to \infty$. Without loss of generality, $Re(z_k) > 0$.

We will eventually replace z_n by $1 + z_n$. Note: If $z = \prod_n (1 + z_n)$ then $z_n \to 0$ as $n \to \infty$.

$$\log(Re^{i\theta}) = \log(r) + i\theta(r > 0, -\pi < \theta < \pi)$$

Note: $z = \prod_n (1 + z_n)$ and $z \notin \mathbb{R}_{\leq 0} \Leftrightarrow \log(z) = \sum_n \log(1 + z_n)$.

Fact: $\prod_{n>1} (1+z_n)$ converges to some complex number note in $\mathbb{R}_{\leq 0}$ iff $\sum_n \log(1+z_n)$ converges in



because log is holomorphic on the cut complex plane.

Recall: $\log(1 + z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$

has radius of convergence 1. (i.e This representation is valid and converges for $|z| < 1$). Without loss of generality we may assume $|z_n| < 1 \forall n$.

For $|z| < 1$,

$$|1 - \frac{\log(1 + z)}{z}| = |\frac{z}{2} - \frac{z^2}{3} + \frac{z^4}{4} - ...|$$

$$= \frac{1}{2}|z - \frac{2z^2}{3} + \frac{2z^3}{4} - ...|$$

$$\geq \frac{1}{2}(|z| + |z|^2 + |z|^3 + ...)$$

$$= \frac{|z|}{2}(\frac{1}{1-|z|})$$

Taking $|z_n| < \frac{1}{2}$, $\frac{|z|}{2} \leq |\log(1+z)| \leq \frac{3}{2}|z|$.

$$\sum_n \frac{|z|}{2} \leq \sum_n |\log(1+z)| \leq \sum_n \frac{3}{2}|z| \tag{1}$$

<u>Fact:</u> Suppose $Re(z_n) > -1$. Then $\sum_{n\geq 0} \log(1+z_n)$ converges absolutely iff $\sum_{n\geq 0} z_n$ converges absolutely.

Proof. This is straight forward from the Comparrison Test and from (1). $\square$

Corollary.
- $\prod(1+z_n)$ *converges* $\Leftrightarrow \sum \log(1+z_n)$ *converges*
- $\sum \log(1+z_n)$ *converges absolutely* $\Leftrightarrow \sum_n z_n$ *converges absolutely.*

Definition. $\prod(1+z_n)$ converges absolutely iff $\sum_n z_n$ converges absolutely.

<u>Note:</u> $\prod_{n=1}^{\infty}(1+z_n)$ converges absolutely, say to z.

$\Rightarrow \log(z) = \sum_{n\geq 1} \log(1+z_n)$ converges to a complex number.

$\Rightarrow \log(z) = a + bi(a \in \mathbb{R}, -\pi <$ ¡$\pi)$.

$\Rightarrow z \neq 0$ since $z = e^a(\cos(b) + i\sin(b)) \neq 0$.

<u>Exercise:</u> Show $\nu(n) \leq \frac{\log(n)}{\log(2)}$.

Proof. Factor $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. Then we have

$$\begin{aligned}
\nu(n) &= \nu(p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}) \\
&= \sum_{i=1}^{k} a_i \\
&\leq \sum_{i=1}^{k} a_i \frac{log(p_i)}{log(2)} \\
&= \frac{log(n)}{log(2)}.
\end{aligned}$$

$\square$

<u>Remark:</u> Recall that $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p (\frac{1}{1-\frac{1}{p^s}}) = \prod_p (1 + \frac{1}{p^s - 1})$ which is valid and converges when $Re(z) > 1$. So,for $Re(z) > 1$, $\zeta(s) = \prod_p (\frac{1}{1-\frac{1}{p^s}}) = \prod_p (1 + \frac{1}{p^s - 1})$ converges and $\zeta(s) \neq 0$

Section 1.4: Average Orders

28 January 2013

Motivation: While we can find upper and lower bounds, at times, the functions we are considering are not well-behaved. This motivates studying the average behavior of arithmetic functions.

Definition. Suppose $f(n)$ is arithmetic and $g(x)$ is monotone increasing. We say that $g(n)$ is the average order of $f(n)$ if

$$\sum_{n \leq x} f(n) = xg(x) + o(xg(x)).$$

Fact 1.4.1 The average order of $d(n)$ is $\log(n)$.

Proof.

$$\begin{aligned}
\sum_{n \leq x} d(n) =& \sum_{n \leq x} \sum_{d|n} 1 \\
=& \sum_{d \leq x} \left[\frac{x}{d}\right] \\
=& \sum_{d \leq x} \left(\frac{x}{d} + \epsilon d\right) \\
=& x \sum_{d \leq x} \frac{1}{d} + O(x) \\
=& x\left(\log(x) + \gamma + O\left(\frac{1}{x}\right)\right) + O(x) \\
=& x \log(x) + O(x)
\end{aligned}$$

Notice:

$$\frac{x}{x \log(x)} \to \frac{1}{\log(x)} \to 0 \quad \text{as} \quad x \to \infty.$$

Thus, $x = o(x \log(x))$. So, we conclude the average order of $d(n)$ is $\log(n)$. $\square$

Fact 1.4.2 There exists $c \in \mathbb{R}$ such that the average order of $\phi(n)$ is cn.

Proof. We start with

$$\begin{aligned}
\sum_{n\le x}\phi(n) &= \sum_{n\le x} n\sum_{d|n}\frac{\mu(d)}{d}\\
&= \sum_{de\le x} e\mu(d)\\
&= \sum_{d\le x}\mu(d)\sum_{e\le \frac{x}{d}} e\\
&= \sum_{d\le x}\mu(d)\left(\frac{\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right]+1\right)}{2}\right)\\
&= \sum_{d\le x}\left(\frac{x^2\mu(d)}{2d^2}+O\left(\frac{x}{d}\right)\right)\\
&=\frac{x^2}{2}\sum_{d\le x}\frac{\mu(d)}{d^2}+O(x\log(x)).
\end{aligned}$$

Let $c=\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}$. This converges absolutely.
Then,

$$\sum_{n\le x}\phi(n)=c\left(\frac{x^2}{2}\right)+\left(-\frac{x^2}{2}\sum_{d\ge x}\frac{\mu(d)}{d^2}\right)+O(x\log(x)).$$

The sum is part of the "tail." Now, let's investigate the tail:

$$\left|\sum_{d\ge x}\frac{\mu(d)}{d^2}\right|\le\int_{x-1}^{\infty}\frac{\mathrm{d}x}{x^2}=o\left(\frac{1}{x}\right).$$

When this term is multiplied by the $\left(-\frac{x^2}{2}\right)$, we have a term that is bounded above by $|x|$ multiplied by a constant. In particular, the entire tail: $\left(-\frac{x^2}{2}\sum_{d\ge x}\frac{\mu(d)}{d^2}\right)=O(x)$. Since this doesn't contribute as much as $O(x\log(x))$, it just gets sucked into that term so that we have

$$\sum_{n\le x}\phi(n)=\frac{cx}{2}(x)+O(x\log(x)).$$

Notice:

$$\frac{x\log(x)}{\frac{cx^2}{2}}\to\frac{2\log(x)}{cx}\to 0 \text{ as } x\to\infty.$$

Thus, the average order of $\phi(n)=\frac{cn}{2}$. $\square$

Exercise 1.4.3:
Show that the average order of $\sigma_1(n) = \sum_{d|n} d$ is cn for $c \in \mathbb{R}$.

Proof. Let $de = n$.

$$\begin{aligned}
\sum_{n\leq x} \sigma_1(n) &= \sum_{n\leq x} \sum_{d|n} d \\
&= \sum_{de\leq x} d \\
&= \sum_{e\leq x} \sum_{d\leq \frac{x}{d}} d \\
&= \sum_{e\leq x} \frac{1}{2}\left(\left[\frac{x}{e}\right]\left[\frac{x}{e}\right] + 1\right) \\
&= \sum_{e\leq x} \frac{1}{2}\left(\left(\frac{x}{e} + \epsilon\right)\left(\frac{x}{e} + \epsilon + 1\right)\right) \\
&= \sum_{e\leq x} \left(\frac{x^2}{e^2} + 2\epsilon\frac{x}{e} + \frac{x}{e} + \epsilon^2 + \epsilon\right) \\
&= \sum_{e\leq x} \left(\frac{x^2}{2e^2} + O\left(\frac{x}{e}\right)\right) \\
&= \frac{x^2}{2}\sum_{e\leq x} \frac{1}{e^2} + O(x\log x)
\end{aligned}$$

Let $c = \frac{1}{2}\sum_{e=1}^{\infty}\frac{1}{e^2}$, and we can see this converges absolutely. Thus, $\sum_{n\leq x}\sigma_1(n)\ cx\cdot$ (x). In other words, we have the average order of $\sigma_1(n) = \sum_{d|n} d$ is cn for $c \in \mathbb{R}$. ∎

Exercise 1.4.4:
Show that $\sum_{n\leq x} q_k(n) = c_k x + O\left(x\frac{1}{k}\right)$ where $c_k = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^k}$.

Proof. We will make us of the identity $q_k(n) = \sum_{d^k|n}\mu(d)$ to aid in this proof.

We begin by writing

$$\begin{aligned}\sum_{n\le x} q_k(n) &= \sum_{n\le x}\sum_{d^k|n}\mu(d)\\ &= \sum_{d^k|n}\left[\frac{x}{d^k}\right]\mu(d)\\ &= x\sum_{d^k|n}\frac{\mu(d)}{d^k} - \sum_{d^k|n}\epsilon_d\mu(d),\end{aligned}$$

where $0 \le \epsilon_d < 1$. We easily bound the sum on the right by noting that

$$\begin{aligned}\sum_{d^k|n}\epsilon_d\mu(d) &= \sum_{d|\sqrt[k]{n}}\epsilon_d\mu(d)\\ &\le \sum_{d|\sqrt[k]{n}} 1\\ &= O(x^{1/k}).\end{aligned}$$

Setting $c_k = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^k}$, (which converges for $k \ge 2$), we now have

$$\sum_{n\le x} q_k(n) = x\left(c_k - \sum_{d^k>x}\frac{\mu(d)}{d^k}\right) + O(x^{1/k}).$$

Finally, note that this middle summation is no larger than our existing error term. Indeed,

$$\sum_{d^k>x}\frac{\mu(d)}{d^k} = \sum_{d>\sqrt[k]{x}}\frac{\mu(d)}{d^k} \le \sum_{d>\sqrt[k]{x}}\frac{1}{d^k} \le \int_{\sqrt[k]{x}}^{\infty}\frac{dt}{t^k} = O(x^{1/k-1}).$$

So we may conclude that $\sum_{n\le x} q_k(n) = c_k x + O(x^{\frac{1}{k}})$. $\square$

Chapter 2: Primes in Arithmetic Progression
Section 2.1: Summation Techniques

Theorem (Dirichlet 1937-1840)
When $(a, q) = 1, \exists$ infinitely many primes, p, such that $p \equiv a(\mod q)$.

Theorem (Partial Summation)
Suppose $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{C}, f(t)$ is differentiable on $[x, 1]$ where $x \in \mathbb{R}$. Set $A(t) = \displaystyle\sum_{n \leq x} a_n$ and $A_0 = 0$.
Then, $\displaystyle\sum_{n \leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt.$

Proof. First, suppose $x \in \mathbb{N}$.
Then, start with the left hand side:

$$\begin{aligned}
\sum_{n \leq x} a_n f(n) &= \sum_{n \leq x} (A(n) - A(n-1))f(n) \\
&= \sum_{n \leq x} A(n)f(n) - \sum_{n \leq x-1} A(n)f(n+1) \\
&= A(x)f(x) + \sum_{n \leq x} A(n)(f(n) - f(n+1)) \\
&= A(x)f(x) - \sum_{n \leq x} A(n) \int_n^{n+1} f'(t)\mathrm{d}t \\
&= A(x)f(x) - \sum_{n \leq x-1} \int_n^{n+1} A(t)f'(t)\mathrm{d}t \\
&= A(x)f(x) - \int_1^x A(t)f'(t)(d)t.
\end{aligned}$$

We will finish this proof letting $x \in \mathbb{R}$ next class.

Summation by Parts

30 January 2013

Last time we proved summation by parts for $x \in \mathbb{Z}$. We now complete the proof.

Let $x \in \mathbb{R}$.

$$\sum_{n \le x} a_n f(n) = \sum_{n \le [x]} a_n f(n)$$

$$= A([x])f([x]) - \int_1^{[x]} A(t)f'(t)dt$$

$$= (A(x)f(x) - \int_1^x A(t)f'(t)dt) + [A([x])f([x]) - A(x)f(x) + \int_{[x]}^x A(t)f'(t)dt]$$

noting that $A(t)$ is constant in $([x], x)$ we get

$$= (A(x)f(x) - \int_1^x A(t)f'(t)dt) + [A(x)(f([x] - f(x))) + A(x)(f(x) - f([x]))]$$

$$= a(x)f(x) - \int_1^x A(t)f'(t)dt.$$

Fact 2.1.2 $\sum_{n \le x} \log n = x \log x - x + O(\log x)$

Proof. Take $f(t) = \log t$ (so $f'(t) = \frac{1}{t}$) and $a_n = 1$. Thus $A(t) = [t]$. Then

$$\sum_{n \le x} \log n = \sum_{n \le} a_n f(n)$$

$$= A(x)f(x) - \int_1^x A(t)f'(t)dt$$

$$= [x] \log x - \int_1^x [t]\frac{1}{t}dt$$

$$= x \log x + O(\log x) - \int_1^x (1 + O(\frac{1}{t}))dt$$

$$= x \log x + O(\log x) - x + O(\log x)$$

$$= x \log x - x + O(\log x)$$

□

Exercise 2.1.3 Show that $\gamma := \lim_{x \to \infty} \sum_{n \le x} \frac{1}{n} - \log x$ exists.

Exercises Suppose the prime number theorem i.e. $\sum_{p \le x} \log p = x + o(x)$. Show

1. $\displaystyle\sum_{p\leq x} 1 = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right)$

2. $\displaystyle\sum_{p\leq x} p = \frac{x^2}{\log x} + o\left(\frac{x^2}{\log x}\right)$

3. $\displaystyle\sum_{p\leq x} \frac{1}{p} = \log\log x + o(\log\log x)$.

Exercise 2.1.4: Show $\displaystyle\sum_{n\leq x} d(n) = x\log x + O(x)$.

Proof.

$$\begin{aligned}
\sum_{n\leq x} d(n) &= \sum_{nm\leq x} 1 \\
&= \sum_{m\leq x}\left[\frac{x}{m}\right] \\
&= \sum_{m\leq x}\left(\frac{x}{m}+\epsilon\right) \\
&= \sum_{m\leq x}\frac{x}{m} + O(x) \\
&= x\sum_{m\leq x}\frac{1}{m} + O(x) \\
&= x\log(x) + \gamma x + O(x) \\
&= x\log(x) + O(x).
\end{aligned}$$

□

Fact 2.1.5 Consider $D(a_n, s) = \displaystyle\sum_{n\geq}\frac{a_n}{n^s}$. Suppose $A(x) = \displaystyle\sum_{n\leq x} a_n = O(x^\delta)$ for some $\delta\in\mathbb{R}$. Then, for $Re(s) > \delta$,

$$D(a_n, s) = s\int_1^\infty \frac{A(t)}{t^{s+1}}dt.$$

Hence $D(a_n, s)$ converges for $Re(s) > \delta$.

Proof. Take $f(x) = x^{-s} = e^{-s\log x}$, so $f'(x) = -sx^{-(s+1)} = -\frac{s}{x}e^{-s\log x}$. Then by partial summation

$$\sum_{1\le n\le x}\frac{a_n}{n^s} = A(x)f(x) - \int_a^x A(t)f'(t)dt$$
$$= \frac{A(x)}{x^s} + s\int_1^x A(t)t^{-(s+1)}dt$$
$$= s\int_1^x \frac{A(t)}{t^{s+1}}dt + O(x^{\delta-s}).$$

Note that for $Re(s) > \delta$ we have

$$D(a_n, s) = \lim_{x\to\infty} s\int_1^x \frac{A(t)}{t^{s+1}}dt + O(x^{\delta-s})$$
$$= s\int_1^\infty \frac{A(t)}{t^{s+1}}dt$$

since $\int_1^x \frac{A(t)}{t^{s+1}}dt = O\left(\frac{x^\delta}{x^s}+1\right)$ and for $Re(s) > \delta$, $\frac{x^\delta}{x^s} \to 0$. Thus, when $Re(s) > \delta$, $\int_1^x \frac{A(t)}{t^{s+1}}dt = O(1)$. Hence $D(a_n, s)$ converges for $Re(s) > \delta$. □

Fact 2.1.6

1. $\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx$ where $x = [x] + \{x\}$.
2. Thus $\lim_{s\to 1^+}(s-1)\zeta(s) = 1$.

Proof. Note that $\zeta(s) = D(1, s)$, and $\sum_{n\le x} 1 = [x]$ (i.e. $\delta = 1$). So by previous facts

$$\zeta(s) = s\int_1^\infty \frac{[t]}{t^{s+1}}dt$$
$$= s\int_1^\infty \frac{t-\{t\}}{t^{s+1}}dt$$
$$= s\int_1^\infty \frac{dt}{t^s}dt - s\int_1^\infty \frac{\{t\}}{t^{s+1}}dt$$

For $Re(s) > 1$

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} \frac{t}{t^{s+1}}dt.$$

Thus

$$(s-1)\zeta(s) = s - (s-1)s\int_1^{\infty} \frac{t}{t^{s+1}}dt.$$

So for $Re(s) > 1$ we have

$$0 \leq \int_1^{\infty} \frac{t}{t^{s+1}}dt \leq \int_1^{\infty} \frac{dt}{t^{s+1}} = \frac{1}{s}.$$

Which implies

$$0 \leq s\int_1^{\infty} \frac{t}{t^{s+1}}dt \leq 1.$$

Hence

$$0 \leq (s-1)s\int_1^{\infty} \frac{t}{t^{s+1}}dt \leq (s-1).$$

So

$$\lim_{s\to 1^+}(s-1)s\int_1^{\infty} \frac{t}{t^{s+1}}dt = 0$$

and therefore

$$\lim_{s\to 1^+}(s-1)\zeta(s) = 1.$$

□

Euler Maclauren Summation

01 February 2013

Fact 2.1.7 $F(x,t) = \sum b_r(x)\dfrac{t^m}{r!} = \dfrac{te^{xt}}{e^t-1}$ where $\{b_r(x)\}_{r=0}^{\infty}$ is defined recursively as $b_0(x) = 1$, $b_r'(x) = rb_{r-1}(x)$, and $\displaystyle\int_0^1 b_r(x)dx = 0$.

Proof.

$$\frac{d}{dt}F(x,t) = \sum_{r\geq 1} b_r'(x)\frac{t^r}{r!} = \sum_{r\geq 1} rb_{r-1}(x)\frac{t^r}{r!} = t\sum_{r\geq 1} b_{r-1}(x)\frac{t^{r-1}}{(r-1)!} = tF(x,t)$$

$$\implies \frac{d}{dt}\log F(x,t) = t$$

So $\log F(x,t) = tx + C(t)$ and hence $F(x,t) = e^{C(t)}e^{xt}$. By definition of $b_r's$,

$$\int_0^1 F(x,t)d = \int_0^1 \sum_{r=0}^{\infty} b_r(x)\frac{t^r}{r!}dx = 1$$

$$\implies 1 = \int_0^1 F(x,t)dx = \int_0^1 e^{C(t)}e^{xt}dx = e^{C(t)}\frac{1}{t}e^{xt}|_{x=0}^1 = e^{C(t)}\frac{e^t-1}{t}$$

$$\implies e^{C(t)} = \frac{t}{e^t-1}.$$

Thus $F(x,t) = \dfrac{te^{xt}}{e^t-1}$. □

Definition. $b_r(x)$ - Bernoulli polynomial
$B_r(x) = b_r(\{x\})$ - Bernoulli function
$B_r = B_r(0) = b_r(0)$ - Bernoulli number

Exercise 2.1.8 $B_{2r+1} = 0, r \geq 1$
Hint: Show that $\dfrac{t}{2} + \displaystyle\sum_{r=0}^{\infty} b_r(0)\frac{t^r}{r!}$ is an even function.

Proof. First note that

$$\frac{t}{e^t-1} + \frac{t}{2} = \frac{t}{2}\frac{e^t+1}{e^t-1}.$$

If we replace t with $-t$, we get

$$\frac{-t}{2}\frac{e^{-t}+1}{e^{-t}-1} = \frac{-t}{2}\frac{1+e^t}{1-e^t} = \frac{t}{2}\frac{e^t+1}{e^t-1}$$

so thus $\dfrac{t}{e^t-1} + \dfrac{t}{2}$ is an even function. If we expand this as the power series

$$\frac{t}{2} + \sum_{r=0}^{\infty} b_r(0)\frac{t^r}{r!},$$

then we see from the following lemma that the coefficients for the odd powers of t must vanish from which our result follows. □

Lemma Let $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$. If f is even, then $a_n = 0$ for n odd.

Proof. Note that we may write

$$f(x) = \sum_{n=0}^{\infty} a_{2n}x^{2n} + x\sum_{n=0}^{\infty} a_{2n+1}x^{2n}.$$

Then if f is even, we must have

$$\sum_{n=0}^{\infty} a_{2n}x^{2n} + x\sum_{n=0}^{\infty} a_{2n+1}x^{2n} = f(x) = f(-x) = \sum_{n=0}^{\infty} a_{2n}x^{2n} - x\sum_{n=0}^{\infty} a_{2n+1}x^{2n}.$$

From this, we see that

$$\sum_{n=0}^{\infty} a_{2n+1}x^{2n}$$

is identically zero so the lemma follows from the uniqueness of power series. □

Remark A similar argument shows that if f is odd, all the coefficients for the even powers of x must vanish.

$$b_1(x) = x - \frac{1}{2}$$

$$b_2(x) = x^2 - x - \frac{1}{6}$$

$$b_3(x) = x^3 - \frac{1}{2}x^2 + \frac{1}{2}x$$

Euler-Maclauren Summation

Let $a, b \in \mathbb{Z}$. We use the Stieltjes integral with respect to the measure $d[t]$.

$$d[x]([x_i, x_{i+1}]) = [x_{i+1}] - [x_i]$$

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)d[t]$$

Note

$$[t] = t - \{t\} \implies [t] = t - B_1(t) - \frac{1}{2} \implies d[t] = dt - dB_1$$

where

$$dB_1([x_i, x_{i+1}]) = B_1(x_{i+1}) - B_1(x_i).$$

Thus

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)dt - \int_a^b f(t)dB_1.$$

Note

$$\int_a^b f(t)dB_1 = fB_1|_a^b - \int_a^b B_1(t)f'(t)dt = [f(b)B_1(0) - f(a)B_1(0)] - \int_a^b B_1(t)f'(t)dt =$$

$$B_1(f(b) - f(a)) - \int_a^b B_1(t)f'(t)dt$$

Recall $B_2'(t) = 2B_1(t)$. Then

$$\sum_{a<n\le b} f(n) = \int_a^b f(t)dt - [f(b)-f(a)]B_1 + \frac{1}{2}\int_a^b f'(t)dB_2(t) =$$

$$\int_a^b f(t)dt - [f(b)-f(a)]B_1 + \frac{1}{2}[f'B_2|_a^b - \int_a^b B_2(t)f''(t)dt].$$

Repeating this process, we obtain Euler-MacLauren summation formula.

Theorem. *Let k be a nonnegative integer and suppose f is $(k+1)$ times differentiable on $[a,b]$ with $a,b \in \mathbb{Z}$. Then*

$$\sum_{n=a}^{b} f(n) = \int_a^b f(t)dt + \sum_{r=0}^{k} \frac{(-1)^{r+1}}{(r+1)!}[f^{(r)}(b) - f^{(r)}(a)]B_{r+1} + \frac{(-1)^k}{(k+1)!}\int_a^b B_{k+1}(t)f^{(k+1)}(t)dt.$$

Proof. The base case has been shown above. Suppose the claim holds for a nonnegative integer $k-1$. Since $B_{k+1}'(t) = (k+1)B_k(t)$, we can write

$$\int_a^b B_k(t)f^{(k)}(t)dt = \frac{1}{k+1}\int_a^b f^{(k)}(t)dB_{k+1}.$$

Now, integrating by parts using $u = f^{(k)}$ and $dv = dB_{k+1}$, we obtain

$$\begin{aligned}\int_a^b f^{(k)}(t)dB_{k+1} &= f^{(k)}(t)B_{k+1}(t)\Big|_a^b - \int_a^b B_{k+1}(t)f^{(k+1)}(t)dt \\ &= B_{k+1}(f^{(k)}(b) - f^{(k)}(a)) - \int_a^b B_{k+1}(t)f^{(k+1)}(t)dt.\end{aligned}$$

Now, using the induction hypothesis and substitution, we confirm the induction claim by writing

$$\begin{aligned}\sum_{n=a}^{b} f(n) &= \int_a^b f(t)dt + \sum_{r=0}^{k-1} \frac{(-1)^{r+1}}{(r+1)!}[f^{(r)}(b) - f^{(r)}(a)]B_{r+1} + \frac{(-1)^{k-1}}{(k)!}\int_a^b B_k(t)f^{(k)}(t)dt \\ &= \int_a^b f(t)dt + \sum_{r=0}^{k-1} \frac{(-1)^{r+1}}{(r+1)!}[f^{(r)}(b) - f^{(r)}(a)]B_{r+1} \\ &\qquad + \frac{(-1)^{k-1}}{(k)!}\cdot\frac{1}{k+1}\left(B_{k+1}(f^{(k)}(b) - f^{(k)}(a)) - \int_a^b B_{k+1}(t)f^{(k+1)}(t)dt\right) \\ &= \int_a^b f(t)dt + \sum_{r=0}^{k} \frac{(-1)^{r+1}}{(r+1)!}[f^{(r)}(b) - f^{(r)}(a)]B_{r+1} + \frac{(-1)^k}{(k+1)!}\int_a^b B_{k+1}(t)f^{(k+1)}(t)dt.\end{aligned}$$

$\square$

Exercise (2.1.10)**.** *Show that*

$$\sum_{n\le x}\frac{1}{n}=\log(x)+\gamma+\frac{1}{2x}+\frac{1}{12x^2}+O\left(\frac{1}{x^3}\right).$$

Proof. blah $\square$

Exercise (2.1.11)**.** *Show, using Euler-MacLauren summation, that*

$$\sum_{n\le x}\frac{1}{\sqrt{n}}=2\sqrt{x}+B+O\left(\frac{1}{\sqrt{x}}\right),$$

where B is some constant.

Proof. Take $f(t)=1/\sqrt{t}$, $a=1$, $b=x$, and $k=0$. Then using Euler-MacLauren summation, we have

$$\sum_{n\le x}\frac{1}{\sqrt{n}}=\int_1^x\frac{1}{\sqrt{t}}dt+\frac{1}{2}\left(\frac{1}{\sqrt{x}}-1\right)-\frac{1}{4}\int_1^x B_1(t)t^{-3/2}dt.$$

Observe that this last integral indeed converges, as we can write it as

$$\begin{aligned}\int_1^x B_1(t)t^{-3/2}dt&=\int_1^\infty B_1(t)t^{-3/2}dt-\int_x^\infty B_1(t)t^{-3/2}dt\\&=C+O\left(\frac{1}{\sqrt{x}}\right).\end{aligned}$$

Thus we have

$$\begin{aligned}\sum_{n\le x}\frac{1}{\sqrt{n}}&=2(\sqrt{x}-1)+\frac{1}{2}\left(\frac{1}{\sqrt{x}}-1\right)-\frac{1}{4}\left(C+O\left(\frac{1}{\sqrt{x}}\right)\right)\\&=2\sqrt{x}+B+O\left(\frac{1}{\sqrt{x}}\right).\end{aligned}$$

$\square$

Exercise (2.1.12)**.** *Let z be a non-zero complex number, and let $\delta>0$ so that $|\arg(z)|<\pi-\delta$. Show that*

$$\sum_{j=0}^{n}\log(z+j)=\left(z+n+\frac{1}{2}\right)\log(z+n)-n-\left(z+\frac{1}{2}\right)\log(z)+\int_0^n\frac{B_1(x)}{z+x}dx.$$

Proof. Take $a = 0$, $b = n$, $f(t) = \log(z+t)$, and $k = 0$. Then using Euler-MacLauren summation, we obtain

$$\begin{aligned}
\sum_{j=0}^{n} \log(z+j) &= \int_0^n \log(z+t)dt + \frac{1}{2}\left(\log(z+n) - \log z\right) + \int_0^n \frac{B_1(t)}{z+t}dt \\
&= (z+n)\log(z+t) - (z+n) - z\log z + z + \frac{1}{2}\left(\log(z+n) - \log z\right) + \int_0^n \frac{B_1(t)}{z+t}dt \\
&= \left(z+n+\frac{1}{2}\right)\log(z+n) - n - \left(z+\frac{1}{2}\right)\log z + \int_0^n \frac{B_1(t)}{z+t}dt,
\end{aligned}$$

where differentiating $\log(z+t)$ is valid because $|\arg(z)|$ stays away from the principal branch cut. $\square$

2.2: Characters (mod q)

04 February 2013

Definition. A *character mod q* is a group homomorphism $\chi\colon (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$.

Recall from group theory that $|(\mathbb{Z}/q\mathbb{Z})^*| = \phi(q)$, and every group element raised to this power is the identity. This means that for any $a \in (\mathbb{Z}/q\mathbb{Z})^*$, we can write

$$1 = \chi(1) = \chi(a^{\phi(q)}) = (\chi(a))^{\phi(q)}.$$

Thus, the image of a character is really just a $\phi(q)^{th}$ root of unity.

Definition. We denote the n^{th} *roots of unity* by $\mu_n = \{z \in \mathbb{C}\colon z^n = 1\}$.

Remark. We can now revise our definition of a character to be any group homomorphism $\chi\colon (\mathbb{Z}/q\mathbb{Z})^* \to \mu_{\phi(q)}$.

We make a few brief observations about our new definitions.

1. Set $\xi_n = e^{2\pi i/n}$ and observe that $\xi_n^n = e^{2\pi i} = 1$, so $\xi_n \in \mu_n$.
2. In fact, we can show that $\mu_n = \{\xi_n^m \colon 0 \le m \le n-1\}$.
   Let $z = re^{i\theta}$ be any element of μ_n. Then for $z^n = 1$, we must have

   $$r = 1 \text{ and } \theta = \left(\frac{2\pi m}{n}\right).$$

   Using the division algorithm, write $m = na + b$ for some $0 \le b < n$. Then

   $$\theta = \left(\frac{2\pi m}{n}\right) = a(2\pi) + \left(\frac{2\pi b}{n}\right),$$

   and so taking all possible values of b yields all n roots of unity.
   For our purposes, this implies that

   $$\mu_{\phi(q)} = \{\xi_{\phi(q)}^m \colon 0 \le m \le \phi(q)\}.$$

3. From group theory, we know that

$$|\xi_n^a| = \frac{|\xi_n|}{(a, |\xi_n|)} = \frac{n}{(a,n)},$$

so when $(a,n) = 1$, then $\mu_n =< \xi_n >=< \xi_n^a >$.

Definition. A *Dirichlet character (mod q)* is the natural domain extension of a character to all of $\mathbb{Z}$. That is, it is the extension $\chi\colon \mathbb{Z} \to \mu_{\phi(q)}$ given by

$$\chi(n) = \begin{cases} \chi([n]_q) & \text{if } (q,n) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Exercise (2.2.1)**.** *Show that χ is completely multiplicative.*

Proof. If $(mn, q) = 1$, then $(m, q) = 1$ and $(n, q) = 1$. So

$$\chi(mn) = \chi([mn]_q) = \chi([m]_q[n]_q) = \chi([m]_q)\chi([n]_q) = \chi(m)\chi(n).$$

If $(mn, q) \neq 1$, then either $(m, q) \neq 1$ or $(n, q) \neq 1$. In either case,

$$\chi(mn) = 0 = \chi(m)\chi(n).$$

□

Definition. A *Dirichlet L-series* is an infinite series of the form

$$L(s, \chi) = D(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Note that when $Re(s) > 1$, then $L(s, \chi)$ converges absolutely.

Exercise (2.2.2)**.** *For $Re(s) > 1$, show that*

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Proof. Since χ is completely multiplicative, we can write

$$\begin{aligned} L(s, \chi) &= \prod_p \left(\sum_{k=0}^{\infty} \frac{\chi(p^k)}{p^{ks}}\right) \\ &= \prod_p \left(\sum_{k=0}^{\infty} \left(\frac{\chi(p)}{p^s}\right)^k\right). \end{aligned}$$

When $Re(s) > 1$, the inside sum converges, so recognizing this geometric series we have

$$
\begin{aligned}
L(s,\chi) &= \prod_p \left(\frac{1}{1 - \frac{\chi(p)}{p^s}} \right) \\
&= \prod_p \left(1 - \frac{\chi(p)}{p^s} \right)^{-1}.
\end{aligned}
$$

$\square$

Definition. The *trivial Dirichlet character (mod q)*, denoted $\chi_0 \colon \mathbb{Z} \to \mu_{\phi(q)}$, is given by

$$
\chi_0(n) = \begin{cases} 1 & \text{if } (q,n) = 1 \\ 0 & \text{otherwise.} \end{cases}
$$

Fact. *If χ, ψ are Dirichlet characters (mod q), then so are $\chi\psi$ and $\overline{\chi}$, where*

$$
\chi \cdot \psi(n) = \chi(n)\psi(n) \text{ and } \overline{\chi}(n) = \overline{\chi(n)}.
$$

Note also that $\chi\overline{\chi}(n) = \chi(n)\overline{\chi(n)} = 1$ if $(n,q) = 1$, and 0 otherwise. In other words, $\chi\overline{\chi} = \chi_0$. Recognizing that complex multiplication is associative and commutative, we have shown that the set of Dirichlet characters forms an Abelian group under multiplication.

Furthermore, because both $(\mathbb{Z}/q\mathbb{Z})^*$ and $\mu_{\phi(q)}$ are finite groups, we know the set of characters mod q is also finite. Therefore, the set of Dirichlet characters is actually a finite Abelian group under multiplication.

To learn more about the structure of $(\mathbb{Z}/q\mathbb{Z})^*$, consider factoring $q = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$. By the Chinese Remainder Theorem, we know that

$$
\mathbb{Z}/q\mathbb{Z} \cong \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k^{a_k}\mathbb{Z},
$$

but moreover, for our purposes, we have

$$
(\mathbb{Z}/q\mathbb{Z})^* \cong (\mathbb{Z}/p_1^{a_1}\mathbb{Z})^* \times \ldots \times (\mathbb{Z}/p_k^{a_k}\mathbb{Z})^*.
$$

So to understand the structure of $(\mathbb{Z}/q\mathbb{Z})^*$, we should really examine the structure of $(\mathbb{Z}/p^a\mathbb{Z})^*$ for prime p.

Fact (2.2.3)**.** *Recall from elementary group theory that $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic.*

Proof. Knowing that $(\mathbb{Z}/p\mathbb{Z})^*$ is finite, let $d_1, d_2, \ldots, d_r$ be a list of distinct orders of the elements in $(\mathbb{Z}/p\mathbb{Z})^*$. Set $e = lcm(d_1, d_2, \ldots, d_r)$ and factor $e = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$. For each $p_j^{a_j}$, there must be some d_i such that

$$
p_j^{a_j} \mid d_i.
$$

So $d_i = p_j^{a_j} t_j$, where $p_j \not| t_j$. Therefore, there is some element in $(\mathbb{Z}/p\mathbb{Z})^*$ - call it x_j - such that
$$|x_j| = d_i = p_j^{a_j} t_j.$$
In particular then, $|x_j^{t_j}| = p_j^{a_j}$, so
$$|x_1^{t_1} \dots x_k^{t_k}| = p_1^{a_1} \dots p_k^{a_k} = e.$$
Because the order of an element divides the order of the group, we know $e \mid p-1$, and so $e \leq p-1$. On the other hand, however, every element $y \in (\mathbb{Z}/p\mathbb{Z})^*$ satisfies the polynomial $y^e - 1 = 0$. Thus, upon factoring this polynomial with at least $p-1$ roots over the field $(\mathbb{Z}/p\mathbb{Z})^*$, we also conclude that $p-1 \leq e$. Therefore, $e = p-1$, and so $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic.

□

Definition. An element $g \in (\mathbb{Z}/p\mathbb{Z})^*$ of order $p-1$ is called a generator or a *primitive root mod* p.

Exercise (2.2.4)**.** *Suppose $p > 2$ is prime. Then $(\mathbb{Z}/p^a\mathbb{Z})^*$ is cyclic.*

Proof. Hint: Let g be a primitive root mod p and show that either g or $g+p$ is a primitive root mod p^2. Then show that if g is a primitive root mod p^2, then g is in fact a primitive root mod p^k for any integer k.

□

We now only need to take care of the case when $p = 2$. Observe that

$$\begin{aligned}
&(\mathbb{Z}/2\mathbb{Z})^* = \{1\} =< 1 > \\
&(\mathbb{Z}/4\mathbb{Z})^* = \mathbb{Z}/2\mathbb{Z} = \{1,3\} =< 3 > \\
&(\mathbb{Z}/8\mathbb{Z})^* = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{1,3,5,7\} =< 5, -1 > \quad \text{(not cyclic!)}.
\end{aligned}$$

Exercise (2.2.5)**.** *For $a \geq 3$, show that the order of 5 in $(\mathbb{Z}/2^a\mathbb{Z})^*$ is 2^{a-2}.*

Proof. Suppose we knew that for $n \geq 3$,
$$5^{2^{n-3}} \equiv 1 + 2^{n-1} \mod 2^n. \tag{2}$$
Upon squaring both sides of this equivalence, we obtain
$$\begin{aligned}
5^{2^{n-2}} &\equiv (1 + 2^{n-1})^2 \\
&\equiv 1 + 2^n + 2^{2n-2} \\
&\equiv 1 + 2^n(1 + 2^{n-2}) \\
&\equiv 1 \mod 2^n,
\end{aligned}$$

which is the desired result. So we set about proving (1). For $n = 3$, the statement is quickly verified. Suppose the claim holds for some integer $k \geq 3$, that is, there is some integer m so that

$$5^{2^{k-3}} = 1 + 2^{k-1} + 2^k \cdot m.$$

Squaring both sides of this equation yields

$$\begin{aligned} 5^{2^{k-2}} &= (1 + 2^{k-1} + 2^k)^2 \\ &= 1 + 2^k + 2^{2k}m + 2^{k+1}m + 2^{2k-2} + 2^{2k}m^2 \\ &= 1 + 2^k + 2^{k+1}(2^{k-1}m + m + 2^{k-3} + 2^{k-1}m^2) \\ &\equiv 1 + 2^k \mod 2^{k+1}. \end{aligned}$$

And so

$$5^{2^{(k+1)-3}} \equiv 1 + 2^{(k+1)-1} \mod 2^{k+1}.$$

$\square$

Exercise (2.2.6)**.** *Use the previous exercise to conclude that for $a \geq 3$,*

$$(\mathbb{Z}/2^a\mathbb{Z})^* \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{a-2}\mathbb{Z} = < -1 > \times < 5 > .$$

6 February 2013

Fact 2.2.7:

The number of $\{\chi \pmod q\} = \varphi(q)$

Proof.

Write $q = 2^{a_0}p_1^{a_1}...p_k^{a_k}(a_0 \geq 0)$

Recall that

$$(\mathbb{Z}/q\mathbb{Z})^* \cong \bigoplus_{i=1}^{k}(\mathbb{Z}/p_i^{a_i}\mathbb{Z})^* \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/a_0\mathbb{Z}$$

For any $\chi modp)$, we can write

$$\chi = \psi_0\chi_0\chi_1...\chi_k$$

where if $q_i \in (\mathbb{Z}/q\mathbb{Z})^*$ is a generator for $(\mathbb{Z}/p_j^{a_j}\mathbb{Z})^*$ and we take $-1, g_0$ to be generators for the $2-$ part if present then

$$\chi_j = \chi(-1) = 1 \quad \chi_j(g_0) = 1$$
$$\chi_j(g_i) = 1 \text{ if } i \neq j$$
$$\chi_j(g_j) = \chi(g_j)$$

Similarly for χ_0, ψ_0

$\square$

Note:

We can think of χ_i as a character mod$p_j^{a_j}$. This is because

If $a \in \mathbb{Z}/q\mathbb{Z}^*$ then we can write

$$a = (-1)^{r_0}(g_0)^{b_0}(g_1)^{b_1}...(g_k)^{b_k}$$

for since $0 \leq c_0 \leq 1$

$$0 \leq b_0 \leq a_0 - 2$$

$$0 \leq b_i \leq a_i$$

Then

$$\chi(a) = \chi((-1)^{c_0})\chi((g_0)^{b_0} \prod_{j=1}^{k} \chi(q_j)$$
$$= \psi_0(a)\chi_0(a)...\chi_k(a)$$

Note:

1. $(\mathbb{Z}/p_i^{a_i})\mathbb{Z})^* \cong \mathbb{Z}/(p_i)^{a_{i-1}}(p_i)^{-1}\mathbb{Z}$.

   The number of $\{\chi \pmod{p}_i^{a_i}\} = \varphi((p_i)^{a_i})$ because if χ is a character $\pmod{p}_i^{a_i}$, then $\chi(g_j^k) = \chi(g_j)^k$.

   $$\chi_l(g_j) = (l^{\frac{2\pi i}{\varphi(p_i^{a_i})}})^l$$

   covers all characters $\pmod{p}_i^{a_i}$.

2. $\{\chi \pmod{2}^{a_0}\} \cong \{\psi_0\chi_0 | \psi_0(-1) = \chi(-1)\chi_0(5) = \chi(5)\}$

   The number of $\{\chi \pmod{2}^{a_0}\} = 2\dot{2}^{a_2} = \varphi(2^{a_0})$.

   The number of $\{\chi \pmod{p}\} = \varphi(2^{a_0})^{\cdot} \prod_{i=1}^{k} \varphi(p_i^{a_i})$ because $\chi = \psi_0\chi_0\chi_1...\chi_k$.

Fact 2.2.8:
If $x \neq x_0$ then $\sum_{a \pmod{q}} \chi(a) = 0$.

Proof.

Since $\chi \neq \chi_0 \pmod{q}$ there exists $(b, a) = 1$ such that $\chi(b) \neq 1$.

$$s := \sum_{a \pmod{q}} \chi(a) = \sum_{a \pmod{q}} \chi(ab) = \chi(b) \sum_{a \pmod{q}} \chi(q) = \chi(b)$$

Hence $(1 - \chi(b))s = 0$ and $1 - \chi(b) \neq 0$. So $s = 0$.

□

Corollary.

$$\sum_{a \pmod{q}} \chi(a) = \begin{cases} \varphi(q) & \textit{if } \chi = \chi_0 \\ 0 & \textit{otherwise} \end{cases}$$

Fact:

For $q > 1$

$$\sum_{\chi \pmod{q}} \chi(n) = \begin{cases} \varphi(q) & \text{if } n \equiv 1(q) \\ 0 & \text{otherwise} \end{cases}$$

Proof.

If $(n, q) > 1$ then $\chi(n) = 0$ for all $\chi pmodq$. If $m \equiv 1 \pmod{q}$ then $\chi(n) = 1$ for all $\chi \pmod{q}$.

$\sum_{\chi \pmod{q}} \chi(n)$ is equal to $\{\chi \pmod{q}\} = \varphi(q)$.

Suppose $n \neq 1 \pmod{q}$ and $(n, q) = 1$. Write $q = \prod_{i=1}^{k} p_i^{a_i}$.

Write $q = \prod_{i=1}^{k} p_i^{a_i}$. Then $n \neq 1 \pmod{p}_i^{a_i}$ for at least one value of i. Without loss of generality $i = 1$, $\psi_2 = \psi_3 = ... = \psi_k = \psi_0 \pmod{q}$ and $\psi_1(g^k) = (l^{\frac{2\pi i}{\varphi(p_i^{a_i})}})^k$.

Define $\psi = \psi_2 \times \psi_3 \times ... \times \psi_k$.

Then $1 = \psi(n) = (l^{\frac{2\pi i}{\varphi(p_i^{a_i})}})^k$ where $n = g^k \pmod{p}_i^{a_i}$.

$$\Leftrightarrow \varphi(p_1^{a_1})n \mid k \Leftrightarrow k = 0$$
$$\Leftrightarrow n \neq 1 \pmod{p}_i^{a_i}$$

Note:

$$T = \sum_{\chi \pmod{q}} \chi(n) = \sum_{\chi \pmod{q}} \psi\chi(n) = \psi(n)T \quad \Rightarrow (1-\psi_n)T = 0 \;\; and 1\text{-}\psi(n) \neq 0$$
$$\Rightarrow T = 0$$

$\square$

Section 2.3: Dirichlet's Theorem

8 February 2013

Recall:
A way to prove there are infinitely many primes by analyzing the zeta function:
We have $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^2} = \prod_p \frac{1}{1-\frac{1}{p^s}}$. This implies:

$$\begin{aligned}\log\zeta(s) &= -\sum_p \log\left(1-\frac{1}{p^s}\right)\\ &= \sum_p \sum_{n\geq 1} \frac{1}{n(p^s)^2}\\ &= \sum_p \frac{1}{p^s} + \sum_p \sum_{n\geq 2} \frac{1}{np^s}.\end{aligned}$$

Note: Let's focus on the second term of the above expression.
For $\mathrm{Re}(s) > 1$,

$$\sum_p \sum_{n\geq 2} \left|\frac{1}{np^{ns}}\right| = \sum_p \sum_{n\geq 2} \frac{1}{np^{n\mathrm{Re}(s)}}.$$

Furthermore, we may rewrite

$$p^{ns} = p^{n(a+bi)} = \exp(\log(p)(n)(a+bi)) = \exp(an\log(p))\exp(nb\log(p)i).$$

Now, if we consider

$$|p^{ns}| = |1|\exp(an\log(p)) = p^{an} = p^{n\mathrm{Re}(s)}.$$

Thus, we have

$$\sum_p \sum_n \frac{1}{np^{\mathrm{Re}(s)}} \le \sum_p \sum_{n\ge 2} \frac{1}{p^n} \le \sum_p \frac{1}{p^2}\left(\frac{1}{1-\dfrac{1}{p}}\right) = \sum_p \frac{1}{p(p-1)} \le \sum_{n\ge 2}\frac{1}{n^2} < \infty.$$

This means $\displaystyle\sum_p \sum_{n\ge 2} \frac{1}{np^s}$ converges.

So, we have $\displaystyle\log(\zeta(s)) = \sum_p \frac{1}{p^s} + \sum_p \sum_{n\ge 2}\frac{1}{np^{ns}}$.

Since have know $\displaystyle\lim_{s\to 1^+}\zeta(s) = \infty \Rightarrow \lim_{s\to 1^+}\log(\zeta(s)) = \infty \Rightarrow \sum_p \frac{1}{p} = \infty$.

Thus, there are infinitely many primes.

Goal: We are working towards showing if $(a,q)=1$, $\displaystyle\sum_{p\equiv a(q)}\frac{1}{p} = \infty$.

Hecho 2.3.1
$\displaystyle\lim_{s\to 1^+}(\log(L(s,\chi_0))) = \infty$.

Proof. Recall:

$$\chi_0(p) = \begin{cases} 0 & \text{if } p \mid q \\ 1 & \text{if } p \nmid q \end{cases}$$

Now, Consider

$$\begin{aligned} L(s,\chi_0) &= \prod_p \left(1 - \frac{\chi_0(p)}{p^s}\right)^{-1} \\ &= \prod_{p\nmid q}\left(1-\frac{1}{p^s}\right)^{-1} \\ &= \zeta(s)\prod_{p||q}\left(1-\frac{1}{p^s}\right). \end{aligned}$$

We chose these expressions because $\zeta(s)$ gives us all the primes - including those that divide q. We multiplied this by $\displaystyle\prod_{p||q}\left(1-\frac{1}{p^s}\right)$ to cancel out all the terms that divide q so we have equivalent expressions.

Note: for $s=1$, $\displaystyle\prod_{p|q}\left(1-\frac{1}{p}\right) = \frac{\phi(q)}{q}$ (a finite number).

Thus,

$$\lim_{s\to 1^+}(L(s,\chi_0)) = \frac{\phi(q)}{q}\lim_{s\to 1^+}\zeta(s) = \infty \Rightarrow \log(L(s,\chi_0)) = \infty.$$

$\square$

Fact 2.3.2
For $\mathrm{Re}(s) > 1$,

$$\sum_{\chi (\bmod\ q)} \log\left(L(s,\chi)\right) = \phi(q) \sum_{n\geq 1} \sum_{p^n \equiv 1(q)} \frac{1}{np^{ns}} \left(= \phi(q) \left[\sum_{p\equiv 1(q)} \frac{1}{p^s} + \sum_{n\geq 2} \sum_{p^n\equiv 1(q)} \frac{1}{np^s} \right] \right).$$

Proof.

$$\begin{aligned}
L(s,\chi) &= \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \\
\Rightarrow \log\left(L(s,\chi)\right) &= -\sum_p \log\left(1 - \frac{\chi(p)}{p^s}\right) \\
&= \sum_p \sum_{n\geq 1} \left(\frac{\chi(p)}{p^s}\right)^n \\
&= \sum_p \sum_{n\geq 1} \frac{\chi^n(p)}{np^{ns}} \\
&= \sum_p \sum_{n\geq 1} \frac{\chi\left(p^n\right)}{np^{ns}}.
\end{aligned}$$

Now that we have a nice expression to work with, we will consider the sum:

$$\begin{aligned}
\sum_{\chi (\bmod\ q)} \left(L(s,\chi)\right) &= \sum_{\chi (\bmod\ q)} \sum_p \sum_{n\geq 1} \frac{\chi\left(p^n\right)}{np^{ns}} \\
&= \sum_p \sum_{n\geq 1} \frac{1}{np^{ns}} \sum_{\chi} \chi\left(p^n\right) \\
&= \sum_{n\geq 1} \sum_{p^n\equiv 1(q)} \frac{\phi(q)}{np^{ns}} \\
&= \phi(q) \sum_{n\geq 1} \sum_{p^n\equiv 1(q)} \frac{1}{np^{ns}}.
\end{aligned}$$

Notice the rearrangement over the second equality is justified because $\mathrm{Re}(s) > 1$, we have absolute convergence (at the end). Also, the penultimate equality follows from the fact that $\sum_{\chi} \chi\left(p^n\right)$ counts characters. In particular, recall:

$$\sum_{\chi} \chi(p) = \begin{cases} \phi(q) & \text{if } p^n \equiv 1(q) \\ 0 & \text{o/w} \end{cases}$$

Hence, we have the result. $\square$

Notation: we denote the order of p in $(\mathbb{Z}/q\mathbb{Z})^*$ as $\text{order}_q(p)$.

Fact 2.3.3
For $\text{Re}(s) > 1$,

$$\sum_{n\geq 1} \frac{a_n}{n^s} = \prod_{\chi(\ q)} L(s, \chi)$$

then $a_1 = 1$ and $a_n = 0$ for $n \geq 2$.

Proof.

$$\begin{aligned}
\prod_{\chi(\text{mod } q)} L(s, \chi) &= \exp\left(\phi(q) \sum_{n\geq 1} \sum_{p^n \equiv 1(q)} \frac{1}{np^P ns}\right) \\
&= \prod_{n\geq 1} \prod_{p^n \equiv 1(q)} \frac{\phi(q)}{np^{ns}} \\
&= \prod_{n\geq 1} \left[\sum_{k=1}^{\infty} \left(\frac{\phi(q)}{n}\right)^k \left(\frac{1}{k}\right) \left(\frac{1}{p^{nks}}\right)\right]
\end{aligned}$$

Now, let $n = f \cdot \text{order}_q(p)$ so that the above expression equals

$$\prod_p \left(\prod_{f=1} \sum_{k=0}^{\infty} \left(\frac{\phi(q)}{f\text{order}_q(p)}\right)^k \left(\frac{1}{k}\right) \left(\frac{1}{p^{fks\text{order}_q(p)}}\right)\right)$$

Furthermore, we may identify the coefficients.
The important thing to note, though, is that each a_i is nonnegative. □

11 February 2013

Fact (2.3.4)**.** *For a nontrivial character χ mod q, we have that*

$$\left|\sum_{n\leq x} \chi(n)\right| \leq q.$$

Proof. First, write

$$[x] = kq + r, \text{ where } 0 \leq r < q.$$

Recall we showed that $\sum_{n\leq q} \chi(n) = 0$ and so $\sum_{nk\leq q} \chi(n) = 0$ for any k. So we have

$$\sum_{n\leq x} \chi(n) = \sum_{n\leq kq} \chi(n) + \sum_{kq<n\leq kq+r} \chi(n) = \sum_{kq<n\leq kq+r} \chi(n).$$

The final summation is just summing r roots of unity, so

$$\left|\sum_{n\leq x} \chi(n)\right| \leq \sum_{kq<n\leq kq+r} |\chi(n)| \leq q - 1 < q.$$

□

Corollary. *$L(s,\chi)$ converges for $Re(s) > 0$.*

Proof. From Fact 2.1.5, we have

$$L(s,\chi) = s\int_1^\infty \frac{s(t)}{t^{s+1}}dt,$$

where $s(t) = \sum_{n\le t}\chi(n)$. The result follows from the approximation in the previous fact. $\square$

Fact (2.3.5)**.** *For a nontrivial character χ mod q, we can show that*

$$L(1,\chi) \neq 0 \iff L(1,\bar{\chi}) \neq 0.$$

Proof. We show one direction because the other is quite similar. Supposing $L(1,\bar{\chi})$ is nonzero, write

$$\begin{aligned} L(1,\bar{\chi}) &= \lim_{x\to\infty}\sum_{n\ge x}\frac{\bar{\chi}(n)}{n} \\ &= \overline{\lim_{x\to\infty}\sum_{n\ge x}\frac{\chi(n)}{n}} \\ &= \overline{L(1,\chi)}. \end{aligned}$$

If the conjugate of $L(1,\chi)$ is nonzero, then certainly $L(1,\chi)$ is nonzero as well. $\square$

Fact (2.3.6)**.** *The residue of the pole at $s = 1$ of $L(s,\chi_0)$ is $\phi(q)/q$. In other words, we have that*

$$\lim_{s\to1^+}(s-1)L(s,\chi_0) = \frac{\phi(q)}{q}.$$

Proof. Using a number of earlier results, we compute

$$\begin{aligned} \lim_{s\to1^+}(s-1)L(s,\chi_0) &= \lim_{s\to1^+}(s-1)\zeta(s)\prod_{p|q}\left(1-\frac{1}{p}\right) \\ &= \frac{\phi(q)}{q}\lim_{s\to1^+}(s-1)\zeta(s) \\ &= \frac{\phi(q)}{q}. \end{aligned}$$

$\square$

Fact (2.3.7)**.** *If $L(1,\chi) \neq 0$ for all non-trivial characters χ mod q, then*

$$\lim_{s\to1^+}(s-1)\prod_{\chi(q)}L(s,\chi) \neq 0.$$

Moreover, part of Dirichlet's theorem holds under this assumption:

$$\sum_{p\equiv 1(q)} \frac{1}{p} = \infty.$$

Proof. We begin by computing

$$\begin{aligned}
\lim_{s\to 1^+}(s-1)\prod_{\chi(q)} L(s,\chi) &= \lim_{s\to 1^+}(s-1)\frac{\phi(q)}{q}\zeta(s)\prod_{\substack{\chi(q)\\ \chi\neq\chi_0}} L(s,\chi)\\
&= \frac{\phi(q)}{q}\lim_{s\to 1^+}\prod_{\substack{\chi(q)\\ \chi\neq\chi_0}} L(s,\chi)\\
&\neq 0 \text{ by hypothesis.}
\end{aligned}$$

In particular then,

$$\lim_{s\to 1^+}\prod_{\chi(q)} L(s,\chi) = \infty.$$

For the second claim, note that for $\mathrm{Re}(s) > 1$, we can write

$$\begin{aligned}
\prod_{\chi(q)} L(s,\chi) &= \prod_{\chi(q)}\prod_{p}\frac{1}{1-\frac{\chi(p)}{p^s}}\\
&= \exp\left(\sum_{\chi(q)}\sum_{p}\sum_{k\geq 0}\frac{\chi(p^k)}{p^{ks}}\right)\\
&= \exp\left(\sum_{p}\sum_{k\geq 0}\frac{1}{p^{ks}}\sum_{\chi(q)}\chi(p^k)\right)\\
&= \exp\left(\phi(q)\sum_{\substack{p\\ p^k\equiv 1(q)}}\frac{1}{p^{ks}}\right)\\
&= \exp\left(\phi(q)\left[\sum_{\substack{p\\ p\equiv 1(q)}}\frac{1}{p}+\sum_{\substack{p\\ k\geq 2\\ p^k\equiv 1(q)}}\frac{1}{p^{ks}}\right]\right).
\end{aligned}$$

Since this product diverges to infinity as s goes to 1 from the right, and since the second sum is convergent (via comparison to the 2nd degree overharmonic series), we have that the leftover sum diverges, which is the desired result. $\square$

Fact (2.3.8)**.** *Fix $(a,q) = 1$. For any integer n, we have that*

$$\sum_{\chi(q)} \bar{\chi}(a)\chi(n) = \begin{cases} \phi(q) & \text{if } n\equiv a \bmod q\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Recall that $\bar{\chi}(a) = \overline{\chi(a)} = (\chi(a))^{-1}\chi(a^{-1})$. So we have

$$\sum_{\chi(q)} \bar{\chi}(a)\chi(n) = \sum_{\chi(q)} \chi(a^{-1}n) = \begin{cases} \phi(q) & \text{if } n \equiv a \bmod q \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

Exercise (2.3.9)**.** *Fix $(a,q) = 1$ and assuming $L(1,\chi) \neq 0$ for all nontrivial characters χ mod q. Show that*

$$\lim_{s\to 1^+} (s-1) \prod_{\chi(q)} L(s,\chi)^{\bar{\chi}(a)} \neq 0.$$

Then, deduce that

$$\sum_{p\equiv a(q)} \frac{1}{p} = \infty.$$

Proof. In the same vein of (2.3.7), we compute

$$\begin{aligned} \lim_{s\to 1^+} (s-1) \prod_{\chi(q)} L(s,\chi)^{\bar{\chi}(a)} &= \lim_{s\to 1^+} (s-1) L(s,\chi_0)^{\bar{\chi}_0(a)} \prod_{\substack{\chi(q) \\ \chi\neq\chi_0}} L(s,\chi)^{\bar{\chi}(a)} \\ &= \lim_{s\to 1^+} (s-1) L(s,\chi_0) \prod_{\substack{\chi(q) \\ \chi\neq\chi_0}} L(s,\chi)^{\bar{\chi}(a)} \\ &= \frac{\phi(q)}{q} \lim_{s\to 1^+} \prod_{\substack{\chi(q) \\ \chi\neq\chi_0}} L(s,\chi)^{\bar{\chi}(a)} \\ &\neq 0, \text{ by hypothesis.} \end{aligned}$$

In particular, we have

$$\lim_{s\to 1^+} \prod_{\chi(q)} L(s,\chi)^{\bar{\chi}(a)} = \infty.$$

$$\lim_{s\to 1^+} \sum_{\chi(q)} \bar{\chi}(a) \log L(s,\chi) = \infty.$$

For the second claim, observe that (2.3.2) gives us

$$\begin{aligned}
\sum_{\chi(q)} \bar{\chi}(a) \log L(s,\chi) &= \sum_{\chi(q)} \bar{\chi}(a) \sum_{p} \sum_{k\geq 0} \frac{\chi(p^k)}{kp^{ks}} \\
&= \sum_{p} \sum_{k\geq 0} \frac{1}{kp^{ks}} \sum_{\chi(q)} \bar{\chi}(a)\chi(p^k) \\
&= \phi(q) \sum_{\substack{p \\ p^k \equiv a(q)}} \frac{1}{kp^{ks}} \\
&= \phi(q) \left(\sum_{\substack{p \\ p\equiv a(q)}} \frac{1}{p^s} + \sum_{\substack{p \\ p^k\equiv a(q) \\ k\geq 2}} \frac{1}{kp^{ks}} \right).
\end{aligned}$$

Taking $s \to 1^+$ here, the LHS diverges to infinity. Seeing as the second sum on the RHS is convergent (compare to second degree overharmonic series), we must have

$$\sum_{p\equiv a(q)} \frac{1}{p} = \infty.$$

$\square$

Note that the past few results prove Dirichlet's theorem under the very strong hypothesis that χ nontrivial and $L(1,\chi) \neq 0$. The next few results are aimed at proving that this hypothesis always holds. For convenience, we'll set

$$F(s) := \prod_{\chi(q)} L(s,\chi) = \sum_{n\geq 1} \frac{a_n}{n^s},$$

where $a_1 = 1$ and $a_n \geq 0$ for $n \geq 2$.

Fact (2.3.10)**.** *Suppose χ_1 mod q is not real (that is, $\chi_1 \neq \bar{\chi_1}$). Then $L(1,\chi_1) \neq 0$.*

Proof. By the corollary to (2.3.4), $L(1,\chi_1)$ converges. To obtain a contradiction, suppose $L(1,\chi_1) = 0$. Write

$$L(s,\chi_1) = (s-1)g(s,\chi_1),$$

where $g(s,\chi_1)$ is some continuous function when $\mathrm{Re}(s) > 0$ except when $s = 1$. Recall from (2.1.5) and (2.3.4) that

$$L(s,\chi_1) = s\int_1^\infty \frac{s(t)}{t^{s+1}}dt, \text{ where } s(t) = \sum_{n\leq t} \chi_1(n) \text{ and } |s(t)| \leq q.$$

In particular, then, $L(s, \chi_1)$ is absolutely convergent when $Re(s) > 0$ and $L(1, \chi_1)$ is differentiable. Therefore, set

$$g(1, \chi_1) = L'(1, \chi_1) = \lim_{s \to 1^+} \frac{L(s, \chi_1) - L(1, \chi_1)}{s - 1} = \lim_{s \to 1^+} g(s, \chi_1).$$

So $g(s, \chi_1)$ is continuous whenever $Re(s) > 0$. From (2.3.5), we also have that $L(1, \bar{\chi_1}) = 0$, so we could similarly write

$$L(s, \bar{\chi_1}) = (s - 1)g(s, \bar{\chi_1}),$$

where $g(1, \bar{\chi_1}) = L'(1, \bar{\chi_1})$. Then we have

$$\begin{aligned}
\lim_{s \to 1^+} \prod_{\chi(q)} L(s, \chi) &= \lim_{s \to 1^+} L(s, \chi_0) \cdot (s - 1)^2 g(s, \chi_1) g(s, \bar{\chi_1}) \cdot \prod_{\substack{\chi \neq \chi_0 \\ \chi \neq \chi_1 \\ \chi \neq \bar{\chi_1}}} L(s, \chi) \\
&= \phi(q) L'(1, \chi_1) L'(1, \bar{\chi_1}) \prod_{\substack{\chi \neq \chi_0 \\ \chi \neq \chi_1 \\ \chi \neq \bar{\chi_1}}} L(1, \chi) \cdot \lim_{s \to 1^+} (s - 1)^2 \\
&= 0.
\end{aligned}$$

But this product is $F(s)$, which is greater than or equal to one when $Re(s) \to 1^+$. Contradiction. Therefore, $L(1, \chi) \neq 0$. □

Dirichlet's Hyperbolic Method

13 February 2013

Note: I intend to clean this up a bit in the next day or two.
Suppose $f = g * h$. Then

$$f(n) = \sum_{d|n} g(d) h\left(\frac{n}{d}\right) = \sum_{ed = n} g(d) h(e).$$

Theorem Given $f = g * h$, for any $y > 0$,

$$\sum_{n \leq x} f(n) = \sum_{d \leq y} g(d) H\left(\frac{x}{d}\right) + \sum_{d \leq \frac{x}{y}} h(d) G\left(\frac{x}{d}\right) - G(y) h\left(\frac{x}{d}\right)$$

where $G(x) = \sum_{n \leq x} g(n)$ and $H(x) = \sum_{n \leq x} h(n)$.

Proof. Suppose $y > 0$.

$$\begin{aligned}
\sum_{n\le x} f(n) &= \sum_{n\le x}\sum_{de=n} g(d)h(e)\\
&= \sum_{de\le x} g(d)h(e)\\
&= \sum_{de\le x, d\le y} g(d)h(e) + \sum_{de\le x, d<y} g(d)h(e)\\
&= \sum_{d\le y} g(d) \sum_{e\le \frac{x}{d}} h(e) + \sum_{e\le \frac{x}{y}} h(e) \sum_{y<d\le \frac{x}{e}} g(d)\\
&= \sum_{d\le y} g(d) H\left(\frac{x}{d}\right) + \sum_{e\le \frac{x}{y}} h(e)\left[G\left(\frac{x}{e}\right) - G(y)\right]\\
&= \sum_{d\le y} g(d) H\left(\frac{x}{d}\right) + \sum_{d\le \frac{x}{y}} h(d) G\left(\frac{x}{d}\right) - G(y)H\left(\frac{x}{d}\right).
\end{aligned}$$

□

Exercise (2.4.2). *Show* $\sum_{n\le x} \sigma_0(n) = x\log(x) + (2\gamma - 1)x + O(\sqrt{x})$.

Proof. First, notice that $\sum_{n\le x}\sigma_0(n) = \sum_{n\le x}\sum_{d|n} 1 = \sum_{n\le x}\sum_{ed=n} 1$.
With this in mind, let's choose $g(d) = 1, h(e) = 1$ so that $H(x) = G(x) = \sum_{n\le x} 1 = [x]$. Letting $y = \sqrt{x}$, we will apply Dirichlet's Hyperbola Method:

$$\begin{aligned}
\sum_{n\le x}\sigma_0(n) &= \sum_{d\le\sqrt{x}} 1\left[\frac{x}{d}\right] + \sum_{d\le\sqrt{x}} 1\left[\frac{x}{d}\right] - [\sqrt{x}]^2\\
&= 2\sum_{d\le\sqrt{x}}\left[\frac{x}{d}\right] - [\sqrt{x}]^2\\
&= 2\sum_{d\le\sqrt{x}}\left(\frac{x}{d}\right) - x + O\left(\sqrt{x}\right)\\
&= 2x\sum_{d\le\sqrt{x}}\left(\frac{1}{d}\right) - x + O\left(\sqrt{x}\right)\\
&= 2x\left(\frac{1}{2}\log\sqrt{x} + \gamma + O\left(\frac{1}{\sqrt{x}}\right)\right) - x + O\left(\sqrt{x}\right)\\
&= x\log x + (2\gamma - 1)x + O\left(\sqrt{x}\right).
\end{aligned}$$

□

Fact 2.4.3 Suppose $\chi(\mod n)$ is real. Define $f(n) = \chi * 1 = \sum_{d|n} \chi(d)$. Then $f(1) = 1$ and $f(n) \geq 0$ for all n. Also if n is a square, then $f(n) \geq 1$.

Proof. Since χ is multiplicative, f is multiplicative. Write $n = p_1^{a_1} \ldots p_k^{a_k}$. Then

$$f(n) = \prod_{p^\alpha || n} f(p^\alpha) = \prod_{p^\alpha || n} \sum_{k=0}^{\infty} \chi(p)^k = \begin{cases} \alpha + 1 & \chi(p) = 1 \\ 1 & \chi(p) = -1, \alpha \text{ even} \\ 0 & \chi(p) = -1, \alpha \text{ odd} \end{cases}.$$

Thus $f(1) = 1$ and $f(n) \geq 0$ for all n. In addition, if n is a square, then α is always even so the result follows. ☐

Fact 2.4.4 Let $f(n) = \sum_{d|n} \chi(d)$ where $\chi \neq \chi_0$ is a character mod q. Then $\sum_{n \leq x} \frac{f(n)}{\sqrt{n}} = 2L(1, \chi)\sqrt{x} + O(1)$.

Proof. We will apply Dirichlet's hyperbola method with $y = \sqrt{x}$.

$$\text{Note: } \frac{f(n)}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{d|n} \chi(d) = \sum_{d|n} \frac{\chi(d)}{\sqrt{n}} = \sum_{de=n} \frac{\chi(d)}{\sqrt{d}} \frac{1}{\sqrt{e}}.$$

We will take $g(d) = \frac{\chi(d)}{\sqrt{d}}, h(d)\frac{1}{\sqrt{d}}, y = \sqrt{x}$ in Dirichlet's hyperbola method. Notes:

$$H(x) = \sum_{n \leq x} \frac{1}{\sqrt{n}} = 2\sqrt{x} + B + O\left(\frac{1}{\sqrt{x}}\right) \text{ Exercise 2.2.11}$$

$$G(x) = \sum_{n \leq x} \frac{\chi(n)}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{\chi(n)}{\sqrt{n}} - \sum_{n > x} \frac{\chi(n)}{\sqrt{n}} = L\left(\frac{1}{2}, \chi\right) - \sum_{n > x} \frac{\chi(n)}{\sqrt{n}}$$

Use partial summation with $a_n = \chi(n), f(t) = \frac{1}{\sqrt{t}}, f'(t) = -\frac{1}{2t^{3/2}}$. Here $A(t) = \sum_{n \leq t} \chi(n) \leq q$ so

$$\sum_{x < n \leq y} \frac{\chi(n)}{\sqrt{n}} = A(y)f(y) = \int_x^y A(t)f'(t)dt - A(x-1)f(x) = \frac{A(y)}{\sqrt{y}} + \int_x^y \frac{A(t)}{2t^{3/2}} - \frac{A(x)}{\sqrt{x+1}}.$$

Letting $y \to \infty$,

$$\sum_{n > x} \frac{\chi(n)}{\sqrt{n}} << \int_x^{\infty} \frac{dt}{t^{3/2} + \frac{1}{\sqrt{x}}}.$$

Thus $G(x) = L\left(\frac{1}{2}, \chi\right) + O\left(\frac{1}{\sqrt{x}}\right)$. ☐

Lemma For $\epsilon > 0$, $\chi \neq \chi_0$, $\displaystyle\sum_{n \leq x} \frac{\chi(n)}{n^\epsilon} = L(\epsilon, \chi) + O\left(\frac{1}{x^\epsilon}\right)$.

Proof. Exercise $\square$

Now use Dirichlet's hyperbola method to write

$$\begin{aligned}
\sum_{n \leq x} \frac{f(n)}{\sqrt{n}} &= \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} H\left(\frac{x}{d}\right) + \sum_{d \leq \sqrt{x}} \frac{1}{\sqrt{d}} G\left(\frac{x}{d}\right) - G(y)H\left(\frac{x}{y}\right) \\
&= \sum_{d \leq x} \frac{\chi(d)}{\sqrt{d}} \left(2\sqrt{\frac{x}{d}} + O(1)\right) + \sum_{x \leq d} \frac{1}{\sqrt{d}} \left(L\left(\frac{1}{2}, \chi\right) + O\left(\frac{\sqrt{d}}{\sqrt{x}}\right)\right) - \left(L\left(\frac{1}{2}, \chi\right) + O\left(\frac{1}{x^{1/4}}\right)\right) \Big(2x^{1/4} \\
&= 2\sqrt{x} \sum_{d \leq x} \frac{\chi(d)}{\sqrt{d}} + O(1) + L\left(\frac{1}{2}, \chi\right) \sum_{d \leq x} \frac{1}{\sqrt{d}} + O\left(\frac{1}{\sqrt{x}} \sum_{d \leq \sqrt{x}} 1\right) - 2L\left(\frac{1}{2}, \chi\right) x^{1/4}.
\end{aligned}$$

Facts:

$$2\sqrt{x} + B + O(\frac{1}{\sqrt{x}})$$

$$G(x) = L\left(\frac{1}{2}, \chi\right) + O\left(\frac{1}{\sqrt{x}}\right)$$

From Dirichlet's hyperbola method, $y = \sqrt{x}$ so

$$\begin{aligned}
\sum_{n \leq x} \frac{f(n)}{\sqrt{n}} &= \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} H\left(\frac{x}{d}\right) + \sum_{d \leq \sqrt{x}} \frac{1}{\sqrt{d}} G\left(\frac{x}{d}\right) - G\left(\sqrt{x}\right) H(\sqrt{x}) \\
&= \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \left(\frac{2\sqrt{x}}{\sqrt{d}}\right) + B + O\left(\frac{\sqrt{d}}{\sqrt{x}}\right) + \sum_{d \leq x} \frac{1}{\sqrt{d}} \left(L\left(\frac{1}{2}, \chi\right) + O\left(\frac{\sqrt{d}}{\sqrt{x}}\right)\right) - \left(L\left(\frac{1}{2}, \chi\right) + O\left(\frac{1}{x^{1/4}}\right.\right.
\end{aligned}$$

We'll simplify these three terms separately. For the first, observe that we can rewrite this as

$$2\sqrt{x} \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{d} + \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} O(1).$$

Using partial summation on this second bit, we'll take $f(t) = t^{-1/2}$ and $a_d = \chi(d)$. Then it can be shown that

$$\sum_{d \leq \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \ll 1.$$

So the first term reduces to

$$2\sqrt{x} \sum_{d \leq \sqrt{x}} \frac{\chi(d)}{d} + O(1).$$

Regarding the second term, we can use partial summation with $f(t) = 1/\sqrt{t}$ and $a_n = 1$ to show that

$$\sum_{d\leq\sqrt{x}} \frac{1}{\sqrt{d}} = 2x^{1/4} + O(1).$$

So we have

$$\begin{aligned}\sum_{n\leq x} \frac{f(n)}{\sqrt{n}} &= 2\sqrt{x} \sum_{d\leq\sqrt{x}} \frac{\chi(d)}{d} + O(1) + 2x^{1/4} L\left(\frac{1}{2},\chi\right) + O(1) - 2x^{1/4} L\left(\frac{1}{2},\chi\right) + O(1). \\ &= 2\sqrt{x} \sum_{d\leq\sqrt{x}} \frac{\chi(d)}{d} + O(1).\end{aligned}$$

We will soon show in (2.4.6) that $\displaystyle\sum_{d>\sqrt{x}} \frac{\chi(d)}{d} = O\left(\frac{1}{\sqrt{x}}\right)$, so we conclude by writing

$$\begin{aligned}\sum_{n\leq x} \frac{f(n)}{\sqrt{n}} &= 2\sqrt{x}\left(L(1,\chi) + O\left(\frac{1}{x}\right)\right) + O(1) \\ &= 2\sqrt{x}L(1,\chi) + O(1).\end{aligned}$$

. ☐

Fact (2.4.5)**.** *Let χ be a non-trivial, real character mod q. Then $L(1,\chi) \neq 0$.*

Proof. To lead to a contradiction, assume $L(1,\chi) = 0$. Then invoking the previous fact gives

$$\sum_{n\leq x} \frac{f(n)}{\sqrt{n}} = O(1).$$

However, we showed that under these hypotheses, $f(n) \geq 1$ when n is a square. So we have

$$\sum_{n\leq x} \frac{f(n)}{\sqrt{n}} \geq \sum_{m\leq\sqrt{x}} \frac{1}{m},$$

which as $x \to \infty$ is much larger than O(1). Contradiction. So $L(1,\chi) \neq 0$. ☐

This completes Dirichlet's theorem in the most uncermonious of ways.

Exercise (2.4.6)**.** *Show that for a non-trivial, real character χ, we have*

$$\sum_{n>x} \frac{\chi(n)}{n} = O\left(\frac{1}{x}\right).$$

Proof. Recall from (2.1.5) that we may write

$$L(1,\chi) = \int_1^\infty \frac{s(t)}{t^2}\,dt, \text{ where } s(t) = \sum_{n\le t} \chi(n).$$

Then, using our familiar bound $|s(t)| \le q$, we have

$$\begin{aligned}
\sum_{n>x} \frac{\chi(n)}{n} &= L(1,\chi) - \sum_{n=1}^{x} \frac{\chi(n)}{n} \\
&= \int_1^\infty \frac{s(t)}{t^2}\,dt - \sum_{n=1}^{x} \frac{\chi(n)}{n} \\
&\le \int_1^\infty \frac{q}{t^2}\,dt + \frac{1}{x} \\
&\ll \frac{1}{x}.
\end{aligned}$$

□

Exercise (2.4.7)**.** *Let χ be a non-trivial, real character, and let $a_n = \sum_{d|n} \chi(d)$. Show that*

$$\sum_{n\le x} a_n = x + L(1,\chi) + O(\sqrt{x}).$$

Proof. Observe that $a_n = \chi(n) \star 1$. This suggests that we use Dirichlet's hyperbola method, taking $f(n) = a_n$, $g(d) = \chi(d)$, $h(d) = 1$, and $y = \sqrt{x}$. Doing this, we have

$$\sum_{n\le x} a_n = \sum_{d\le\sqrt{x}} \chi(d) \left[\frac{x}{d}\right] + \sum_{d\le\sqrt{x}} G\left(\frac{x}{d}\right) - G(\sqrt{x})[\sqrt{x}].$$

Recall our convenient bound, $|G(x)| \le q$. Thus, using this and the result from the previous exercise, we may write

$$\begin{aligned}
\sum_{n\le x} a_n &\le x \sum_{d\le\sqrt{x}} \frac{\chi(d)}{d} + \sum_{d\le\sqrt{x}} q + q[\sqrt{x}] \\
&= x \sum_{d\le\sqrt{x}} \frac{\chi(d)}{d} + O(\sqrt{x}) \\
&= x\left(L(1,\chi) + O\left(\frac{1}{\sqrt{x}}\right)\right) + O(\sqrt{x}) \\
&= xL(1,\chi) + O(\sqrt{x}).
\end{aligned}$$

□

Exercise (2.4.8)**.** *Use the previous exercise to construct an alternate proof that $L(1,\chi) \ne 0$ for any real, non-trivial character χ.*

Proof. As before, to reach a contradiction, assume $L(1, \chi) = 0$. Then from the previous exercise,

$$F(x) = \sum_{n \leq x} a_n = O(\sqrt{x}).$$

Consider the Dirichlet series, $D(a_n, s)$. Recall the result from (2.1.5), namely that for $F(x) = O(x^{1/2})$, we have that

$$D(a_n, s) = s \int_1^\infty \frac{F(t)}{t^{s+1}} dt,$$

which converges whenever $\mathrm{Re}(s) > 1/2$. Note also that

$$D(a_n, s) = L(s, \chi) L(s, 1) = L(s, \chi) \zeta(s).$$

On the RHS, we know that $L(s, \chi)$ is defined for $\mathrm{Re}(s) > 0$, and we showed in (2.1.6) that $\zeta(s)$ has meromorphic continuation to $\mathrm{Re}(s) > 0$. So write $s = 1/2 + \epsilon$ for some $0 < \epsilon < 1/2$. Then

$$\lim_{\epsilon \to 0^+} D(a_n, s) = \lim_{\epsilon \to 0^+} L(1/2 + \epsilon, \chi) \zeta(1/2 + \epsilon) = L(1/2, \chi) \zeta(1/2) \in \mathbb{C}.$$

But from (2.4.3), we also have that

$$D(a_n, s) = \sum_{m=1}^\infty \frac{a_m}{m^{1/2+\epsilon}} \geq \sum_{m=1}^\infty \frac{1}{m^{1+2\epsilon}} = \zeta(1 + 2\epsilon),$$

and $\lim_{\epsilon \to 0^+} \zeta(1 + 2\epsilon) = \zeta(1) = \infty$. Contradiction. So $L(1, \chi) \neq 0$. □

18 February 2013

Chapter 3: Prime Number Theorem

Definition.

$\pi(x) =$ the number of $\{p < 1 \mid p \text{ is prime}\}$

Prime Number Theorem (1896)

$$\pi(x) \sim \frac{x}{\log(x)} + O(x^{\frac{1}{2}+\varepsilon})$$

1 3.1. Chebysev's Theorem

Observation:

$$\binom{2n}{n} = \frac{(2n)(2n-1)...(n+1)}{n(n-1)...2.1}$$

is divisible by all primes $n \leq p \leq 2n$.

Fact 3.1.1

Letting $\theta(n) = \sum_{p \leq n} \log(p)$.

Then $\theta(n) \leq 4\log(2)n$.

Proof. Note

$$\sum_{n<p\leq 2n} \log(p) \leq \sum_{p|\binom{2n}{n}} \log(p) \leq \log\binom{2n}{n}$$

$$\binom{2n}{n} \leq \sum_{j=0}^{2n} \binom{2n}{j} = (1+1)^{2n} = 2^{2n}$$

$$\theta(2n) - \theta(n) \leq \log(\binom{2n}{n}) \leq 2n\log(2)$$

$$\Rightarrow \theta(2^{r+1}) = \theta(2^r) \leq 2^{r+1}\log(2)$$

So

$$\theta(2^{m+1}) = \theta(2^{m+1}) - \theta(2^0)$$

$$= \sum_{r=0}^{m} [\theta(2^{r+1}) - \theta(2^r)]$$

$$\leq \sum_{r=0}^{n} 2^{r+1}\log(2) = 2\log(2)\sum_{r=0}^{m} 2^r$$

$$= 2\log(2)(2^{m+1} - 1)$$

So, for $2^m \leq n \leq 2^{m+1}$

$$\theta(m) = \theta(2^m) + (\theta(n)) - \theta(2^m))$$

$$= 2^{m+1}\log(2) + (\theta(2^{m+1}) - \theta(2^m))$$

$$\leq 2^{m+1}\log(2) + 2^{m+1}(\log(2))$$

$$= 2^{m+2}\log(2) = 4 \cdot 2^{n}\log(2)$$

$$\leq 4\log(2)n$$

□

Exercise. *Use the previous result to deduce that*

$$\pi(n) = \sum_{p\leq n} 1 \leq 4\log 2 \cdot \frac{n}{\log n}.$$

Proof. Observe that

$$\pi(n) = \sum_{p\leq n} 1 = \sum_{p\leq n} \frac{\log p}{\log p} \leq \frac{1}{\log n}\sum_{p\leq n}\log p = \frac{\theta(n)}{\log n}.$$

So from the previous exercise, we have that

$$\pi(n) \leq 4\log 2 \cdot \frac{n}{\log n}.$$

□

Exercise 3.1.2

Show

$$\theta(n) \leq 2\log(2)n$$

Exercise. *Induct on n to derive the better upper bound*

$$\theta(n) \leq 2\log 2 \cdot n.$$

Proof. It is quickly confirmed that the claim is true for $n = 1, 2, 3, 4, 5$. Take any $n > 5$ and suppose the claim holds for every integer less than n. If n is composite, then using the result from (3.1.1) we have

$$\theta(n) = \theta(n-1) \leq 2(n-1)\log 2 \leq 2n\log 2.$$

If n is prime, write $n = 2m+1$. In the proof of (3.1.1) we showed

$$\theta(2m+1) - \theta(m) \leq \log\binom{2m+1}{m}$$

and also

$$\binom{2m+1}{m} + \binom{2m+1}{m+1} \leq \sum_{j=0}^{2m+1} \binom{2m+1}{j} = 2^{2m+1}.$$

Recalling that the middle binary coefficients of odd powers are equal, this implies

$$\binom{2m+1}{m} \leq 2^{2m}.$$

Thus, using the induction hypothesis and the previous work, we have

$$\begin{aligned}\theta(2m+1) &\leq \log \binom{2m+1}{m} + \theta(m) \\ &\leq 2m \log 2 + 2m \log 2 \\ &\leq 2 \log 2 \cdot (2m+1).\end{aligned}$$

□

Fact 3.1.3

Let $\psi(x) = \sum_{p^a|x} \log(p) = \sum_{n \leq \chi} \lambda(n)$. $(= \theta(x) +$ error$)$

Then

$$lcm[1, 2, ..., n] = e^{\psi(n)}$$

Proof.

$$lcm[1, 2, ..., n] = \prod_{p \leq n} p^{e_p}$$

where $e_p = max_{1 \leq m \leq n}(ord_p(m))$.

$$e*_p = [\log_p(n)] = \lfloor \frac{\log(n)}{\log(p)} = \sum_{p^\alpha \leq n} 1$$

$$\log(lcm[1, 2, ..., n]) = \sum_{p \leq n} e_p \log(p) = \sum_{p \leq n} (\sum_{p^\alpha \leq n} 1) \log(p)$$

$$= \log_{p^\alpha \leq n} \log(p) = \sum_{m \leq n} \lambda(m)$$

□

Fact 3.1.4

Note that $e^{\psi(2n+1)} \int_0^1 x^n(1-x)^n dx \in \mathbb{N}$ which implies $\psi(2n+1) \geq 2 \log(2)n$.

Proof. Note:

$$I = \int_0^1 x^n(1-x)^n dx$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \int_0^1 x^{n+k} dx$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{1}{n+k+1}$$

which is clearly rational.

$\square$

Recall:

$$e^{\psi(2n+1)} = lcm(1, 2, ..., 2n+1)$$

$$\Rightarrow e^{\psi(2n+1)} I \in \mathbb{N}$$

$$e^{\psi(2n+1)} I \geq 1$$

$$e^{\psi(2n+1)} \geq \frac{1}{I}$$

Notice $x(1-x)$ is maximized at $x = \frac{1}{2}$, i.e $x = \frac{1}{2}$. $\frac{1}{2}(1 - \frac{1}{2} = \frac{1}{4}$. So the max value of $x^n(1-x)^n$ on $[0,1]$ is $\frac{1}{2^{2n}}$.

$$I < \frac{1}{2^{2n}}$$

$$e^{\psi(2n+1)} \geq \frac{1}{I} > 2^{2n}$$

$$\Rightarrow \psi(2n+1) > 2\log(2)n$$

Fact 3.1.5

There is $A, B \in \mathbb{R}_{>0}$ such that $\frac{AX}{\log(x)} \leq \pi(x) \leq \frac{Bx}{\log(x)}$.

Proof. Recall $\theta(x) \le 2\log(2)x$.

$$\psi(2n+1) \ge 2\log(2)n$$

$$\Rightarrow \psi(2n+2) \ge (2n+1)\log(2)$$

$$\psi(x) \ge \log(2)x$$

So

$$\sum_{\sqrt{x}<p<x} \log(p) \le 2x\log(2)$$

$$\pi(x) - \pi(\sqrt{x}) \le 2\log(2)x$$

$$\pi(x) \le \frac{4x\log(2)}{\log(x)} + \pi(\sqrt{x})$$

$$\le 4x\frac{\log(2)}{\log(x)} + \sqrt{x}$$

$$\pi(x) << \frac{x}{\log(x)}$$

□

Prime Counting Functions

20 February 2013

Definitions

1. $\Pi(x) = \sum_{p\le x} 1$
2. $\Theta(x) = \sum_{p\le x} \log p$
3. $\Psi(x) = \sum_{n\le x} \Lambda(n)$

Exercise. *Show that* $\psi(x) = \theta(x) + O(\sqrt{x}\log^2 x)$.

Proof. Notice that

$$\psi(x) = \theta(x) + \sum_{\substack{p^\alpha \leq x \\ \alpha \geq 2}} \log(p),$$

and so it suffices to show that this sum is of order $\sqrt{x}\log^2 x$. Observe that

$$\begin{aligned}
\sum_{\substack{p^\alpha \leq x \\ \alpha \geq 2}} \log(p) &\leq \sum_{\substack{p^\alpha \leq x \\ \alpha \geq 2}} \log x \\
&\leq \log x \sum_{p^\alpha \leq x} 1 \\
&\leq \log x \sum_{p \leq \sqrt{x}} \sum_{\alpha \leq \frac{\log x}{\log p}} 1 \\
&\leq \log^2 x \sum_{p \leq \sqrt{x}} 1 \\
&\leq \sqrt{x}\log^2 x.
\end{aligned}$$

□

Facts

1. $\Psi(x) = \Theta(x) = O\left(\sqrt{x}\log x\right)$

2. $\Pi(x) = O(\sqrt{x})$

3. $\Theta(x) \ll \sqrt{x}\log x$

4. $\Psi(x) \ll \sqrt{x}\log x$

5. $\Psi(x) \gg x$

Proof. Recall $e^{\Psi(x)}\int_0^1 x^n(1-x)^n \mathrm{d}x \in \mathbb{N} \Rightarrow e^{\Psi(x)} \leq \frac{1}{\mathrm{I}}$ where I is bounded above so $\frac{1}{\mathrm{I}} \leq 2^{2n} \Rightarrow \Psi(2n+1) \leq 2n\log n \gg n$. Select n such that $2n+3 \leq x > 2n+1$, then $\Psi(x) \leq \Psi(2n+1) \gg n \geq \frac{x-3}{2} = \frac{1}{2}x - \frac{3}{2} \ll x$. □

Note

$$\begin{aligned}\Theta(x)-\Theta(x) &= \sum_{\sqrt{x}<p<x} \log p \\ &= \Theta(x) + O(\sqrt{x}\log x) \\ &\gg \Psi(x) + O(\sqrt{x}\log x) \\ &\Rightarrow \log x[\Pi(x)-\Pi(\sqrt{x})] \\ &\geq \sum_{\sqrt{x}<p<x} \log p \\ &\gg x.\end{aligned}$$

This implies

$$\Pi(x)\log x \gg x + O\left(\Pi(x)\log x\right) \gg x$$

so that we can say $\Pi(x) \gg \dfrac{x}{\log x}$. Thus, there exist A, B $\in \mathbb{R}_{>0}$ such that

$$\frac{\mathrm{A}x}{\log x} \leq \Pi(x) \leq \frac{\mathrm{B}x}{\log x}.$$

Challenge: Find A, B that work assuming x sufficiently large.

Exercise (3.1.6)**.** *Use Euler-Maclauren summation to show*

$$T(x) \colon = \sum_{n\leq x} \log n = x\log x - x + C + O\left(\frac{1}{x}\right).$$

Proof. By Euler-Maclauren summation,

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)dt + \sum_{r=0}^{k} \frac{(-1)^{r+1}}{(r+1)!}[f^{(r)}(b)-f^{(r)}(a)]B_{r+1} + \frac{(-1)^k}{(k+1)!}\int_a^b B_{k+1}(t)f^{(k+1)}(t)dt.$$

Take $a=1$, $b=x$, $f(t)=\log t$, and $k=1$ in Euler-Maclauren summation. We'll look at the integral part first. Observe that

$$\begin{aligned}\frac{1}{2}\int_1^x \frac{B_2(t)}{t^2}dt &= \frac{1}{2}\int_1^x \frac{1/6-\{t\}+\{t\}^2}{t^2}dt \\ &= \frac{1}{12}\int_1^x \frac{dt}{t^2} - \frac{1}{2}\int_1^x \frac{t-[t]}{t^2}dt + \frac{1}{2}\int_1^x \frac{\{t\}^2}{t^2}dt \\ &= -\frac{1}{2}\int_1^x \frac{dt}{t} + C + O\left(\frac{1}{x}\right) \\ &= -\frac{1}{2}\log x + C + O\left(\frac{1}{x}\right).\end{aligned}$$

So with the result of the summation formula, we have

$$\begin{aligned}\sum_{n\le x}\log n &= \int_1^x \log t dt + \frac{1}{2}\log x + \frac{1}{12}\left(\frac{1}{x}-1\right) - \frac{1}{2}\log x + C + O\left(\frac{1}{x}\right)\\ &= x\log x - x + C + O\left(\frac{1}{x}\right).\end{aligned}$$

□

Fact 3.1.7
Show $\sum_{d\le x}\frac{\Lambda(d)}{d} = \log x + O(1)$.

Proof. Recall $\sum_{d|n}\Lambda(n) = \log n$.

$$\begin{aligned}T(x) &= \sum_{n\le x}\log n\\ &= \sum_{de\le x}\Lambda(d)\\ &= \sum_{d\le x}\Lambda(d)\left[\frac{x}{d}\right]\\ &= x\sum_{d\le x}\frac{\Lambda(d)}{d} + \sum_{d\le x}\Lambda(d)\left\{\frac{x}{d}\right\}.\end{aligned}$$

So, take 1 as an upper bound on the fractional part yielding
$T(x) = x\sum_{d\le x}\frac{\Lambda(d)}{d} + O(\Psi(x))$ because $\sum\Lambda(d) = \Psi(x)$. This implies:

$$\begin{aligned}\sum_{d\le x}\frac{\Lambda(d)}{d} &= \frac{T(x)}{x} + O(1) && \text{(b/c } \Psi(x) \ll x)\\ &= \log x + O(1) && \text{(by exercise 3.1.6.)}\end{aligned}$$

□

Fact 3.1.8
$\sum_{p\le x}\frac{1}{p} = \log\log x + O(1)$.

Proof.

$$\begin{aligned}
\sum_{n\le x}\frac{\Lambda(n)}{n} &= \sum_{p\le x}\frac{\log p}{p}+\sum_{\substack{p^{\alpha}\le x\\ a\ge 2}}\frac{\log p}{p^{\alpha}}\\
&= \sum_{p\le x}\frac{\log p}{p}+\sum_{\sqrt{x}}\log p\sum_{2\le\alpha\le\frac{\log x}{\log p}}\frac{1}{p^{\alpha}}\\
&= \sum_{p\le x}\frac{\log p}{p}+O\left(\sum_{p\le\sqrt{x}}\log p\left(\frac{1}{p^2}\right)\left(\frac{1}{1-\frac{1}{p}}\right)\right)\\
&= \sum_{p\le x}\frac{\log p}{p}+O\left(\sum_{p\le\sqrt{x}}\frac{\log p}{p(p-1)}\right)\\
&= \sum_{p\le x}\frac{\log p}{p}+O\left(\frac{1}{n^{\frac{2}{3}}}\right)\\
&= \sum_{p\le x}\frac{\log p}{p}+O(1)
\end{aligned}$$

$$\Rightarrow \sum_{p\le x}\frac{\log p}{p}=\sum_{n\le 1}\frac{\Lambda(n)}{n}+O(1).$$

By partial summation, we let $f(t)=\dfrac{1}{\log t}, f'(t)=-\dfrac{1}{t\log^2 t}, a_n=\begin{cases}\dfrac{\log p}{p} & \text{if } n=p\\ 0 & \text{o/w}\end{cases}$

So,

$$\begin{aligned}
\sum_{p\le x}\frac{1}{p} &= \frac{1}{\log x}(\log x+O(1))+\int_2^x\frac{\log t+O(1)}{t\log^2 t}\mathrm{d}t\\
&= O(1)+\int_2^x\frac{\mathrm{d}t}{t\log t}+O\left(\frac{\mathrm{d}t}{t\log^2 t}\right) \qquad (u=\log t)\\
&= O(1)+\int_{\log 2}^{\log x}\frac{\mathrm{d}u}{u}+O\left(\int_{\log 2}^{\log x}\frac{\mathrm{d}u}{u^2}\right)\\
&= \log\log x+O(1)+O\left(\frac{1}{\log x}+\frac{1}{\log 2}\right)\\
&= \log\log x+O(1).
\end{aligned}$$

$\square$

22 February 2013

The following theorem, known as **Bertrand's Postulate** is a famous result in number theory. It was first proved by Chebyshev and later by Erdös. The proof we will follow, however, is Ramanujan's.

Theorem (3.1.9)**.** *There is a prime between n and $2n$ for n sufficiently large.*

Proof. As a preliminary observation, recall that for a monotonic decreasing sequence tending to 0 - denote it (a_n) - it is true that

$$a_0 - a_1 \leq \sum_{n=0}^{\infty}(-1)^n a_n.$$

To begin the proof, note that

$$T(x) = \sum_{n \leq x} \log x = \sum_{de \leq x} \Lambda(d) = \sum_{e \leq x} \psi\left(\frac{x}{e}\right),$$

so in particular

$$\begin{aligned} T(x) - 2T\left(\frac{x}{2}\right) &= \sum_{e \leq x} \psi\left(\frac{x}{e}\right) - 2\sum_{e \leq \frac{x}{2}} \psi\left(\frac{x}{2e}\right) \\ &= \sum_{e \leq x}(-1)^{e-1}\psi\left(\frac{x}{e}\right). \end{aligned}$$

So by the observation above and from (3.1.6), we have that

$$\begin{aligned} \psi(x) - \psi\left(\frac{x}{2}\right) &\leq T(x) - 2T\left(\frac{x}{2}\right) \\ &= x\log x - x + O(\log x) - \left(x\log\frac{x}{2} - x + 2O\left(\log\frac{x}{2}\right)\right) \\ &= x\log 2 + O(\log x). \end{aligned}$$

Rearranging and iterating the same inequality k times, we obtain

$$\begin{aligned} \psi(x) &\leq x\log 2 + O(\log x) + \psi\left(\frac{x}{2}\right) \\ &= x\log 2 + O(\log x) + \left(\frac{x}{2}\log 2 + \psi\left(\frac{x}{4}\right) + O\left(\log\frac{x}{2}\right)\right) \\ &= x\log 2\left(1 + \frac{1}{2}\right) + \psi\left(\frac{x}{4}\right) + 2O(\log x) \\ &\dots \\ &= x\log 2\sum_{n=0}^{k}\left(\frac{1}{2}\right)^n + \psi\left(\frac{x}{2^k}\right) + kO(\log x). \end{aligned}$$

It is important we count the error terms here because our choice of k will depend on x. In fact, we would like $2^k \geq x$, which amounts to choosing

$$k = \left\lceil \frac{\log x}{\log 2} \right\rceil.$$

Doing this yields the nice bound

$$\psi(x) \leq 2x \log 2 + O(\log^2 x).$$

Returning to the altenating series involving ψ, write

$$\psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) \geq T(x) - 2T\left(\frac{x}{2}\right) = x \log 2 + O(\log x).$$

Therefore we can use the bound deduced above to obtain

$$\begin{aligned} \psi(x) - \psi\left(\frac{x}{2}\right) &\geq x \log 2 + O(\log x) - \psi\left(\frac{x}{2}\right) \\ &\geq x \log 2 + O(\log x) - \frac{2 \log 2}{3} x + O(\log^2 x) \\ &= \frac{\log 2}{3} x + O(\log^2 x). \end{aligned}$$

At this point, we use our knowledge of the relationship between $\psi(x)$ and $\theta(x)$ to conclude the proof. We now have that

$$\begin{aligned} \theta(x) - \theta\left(\frac{x}{2}\right) &= \psi(x) - \psi\left(\frac{x}{2}\right) + O(\sqrt{x} \log^2 x) \\ &\geq \frac{\log 2}{3} x + O(\sqrt{x} \log^2 x). \end{aligned}$$

Since x grows quicker than $\sqrt{x} \log^2 x$, the RHS is eventually positive, which means there is some prime between x and $x/2$ whenever x is sufficiently large. $\square$

Remark. A weekend challenge would be to make this theorem effective. That is, determine the constant in the $O(\sqrt{x} \log^2 x)$ term in order to find a lower bound on x. This would involve returning to earlier exercises to establish constants in the appropriate estimates of $\theta(x)$ and $\psi(x)$.

Further Topics in the Theory of Prime Counting Functions

25 February 2013

Fact (3.1.10)**.** *Suppose* $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$. *let* $s(x) = \sum_{n \leq x} a_n$. *If* $\lim_{x \to \infty} \frac{s(x)}{x} = \alpha$, *then* $\sum_{n \leq x} \frac{a_n}{n} = \alpha \log(x) + o(\log(x))$ *as* $x \to \infty$, *i.e.* $\sum_{n \leq x} \frac{a_n}{n} \ \alpha \log(x)$.

Proof. By partial summation with $f(t) = \frac{1}{t}$, $\sum_{n \leq x} \frac{a_n}{n} = \frac{s(x)}{x} + \int_1^x \frac{s(t)}{t^2} dt$. Note that $\lim_{x \to \infty} \frac{s(x)}{x} = \alpha$ implies $s(x) = \alpha x + E(x)$ where $\frac{E(x)}{x} \to \infty$ as $x \to \infty$, i.e. $s(x) = \alpha x + o(x)$. Thus $\int_1^x \frac{s(t)}{t^2} dt = \int_1^y \frac{s(t)}{t^2} dt + \int_y^x \frac{s(t)}{t^2} dt$ where we take $y = y(x)$ such that $y(x) \to \infty$ as $x \to \infty$. Then

$$\int_y^x \frac{s(t)}{t^2} dt = \int_y^x \frac{\alpha t + o(t)}{t^2} dt = \alpha \int_y^x \frac{dt}{t} + o(\int_y^x \frac{dt}{t}) =$$

$$\alpha \log(x) + O(\log(y)) + o(\int_y^x \frac{dt}{t}) = \alpha \log(x) + O(\log(y)) + o(\log(x)).$$

Taking $\log(y) = o(\log(x))$, e.g. $\log(y) = \log(\log(x))$ and thus $y = \log(x)$. Then $\int_y^x \frac{s(t)}{t^2} dt = \alpha \log(x) + o(\log(x))$. □

Note: $\frac{s(x)}{x} \to \infty$ as $x \to \infty$ so there exists $M \in \mathbb{R}_{>0}$ such that $x > M \implies |\frac{s(x)}{\alpha} - \alpha| < 1 \implies |s(x) - \alpha| < x \implies s(x) = O(x)$. So $\int_1^y \frac{s(t)}{t^2} dt = O(\int_1^y \frac{dt}{t}) = O(\log(y))$. Note $\frac{s(x)}{x} = O(1)$. Thus $\sum_{n \le x} \frac{a_n}{n} = \alpha \log(x) + o(\log(x))$ provided that $\log(y) = o(\log(x))$ where $y = y(x) \to \infty$ as $x \to \infty$. Taking $y = \log(x)$ works.

Exercise (3.1.11)**.** *Show that $\psi(x) \sim x$ if and only if $\pi(x) \sim x/\log x$.*
Hint: *($\Rightarrow$) Use partial summation on $\sum_{2 \le n \le x} \frac{\Lambda(n)}{\log(n)}$. Note $\sum_{n \le x} \frac{\Lambda(n)}{\log(n)} = \pi(x) + O(\sqrt{x} \log(x))$. ($\Leftarrow$) Use partial summation on $\sum_{x \le x} f(n) \log(n)$ where $f(n) = \begin{cases} 1 & n \text{ prime} \\ 0 & \text{otherwise} \end{cases}$.*

Proof. For the first direction, keep in mind that by hypothesis we have that $\psi(x) = x + o(x)$ and $\psi(x)/x = O(1)$. Consider the sum

$$\sum_{2 \le n \le x} \frac{\Lambda(n)}{\log n}.$$

Taking $f(t) = 1/\log t$ and $a_n = \Lambda(n)$ yields

$$\begin{aligned} \sum_{2 \le n \le x} \frac{\Lambda(n)}{\log n} &= \frac{\psi(x)}{x} + \int_2^x \frac{\psi(t)}{t \log^2 t} dt \\ &= \frac{x}{\log x} + o\left(\frac{x}{\log x}\right) + O\left(\int_2^x \frac{dt}{\log^2 t}\right). \end{aligned}$$

Observe that

$$\begin{aligned}\int_2^x \frac{dt}{\log^2 t} &= \int_2^{\sqrt{x}} \frac{dt}{\log^2 t} + \int_{\sqrt{x}}^x \frac{dt}{\log^2 t}\\ &\leq \frac{\sqrt{x}}{\log^2 2} + \frac{x-\sqrt{x}}{\log^2 x}\\ &= O(\sqrt{x}) + O\left(\frac{x}{\log^2 x}\right)\\ &= O\left(\frac{x}{\log^2 x}\right).\end{aligned}$$

Therefore, we have

$$\sum_{2\leq n\leq x} \frac{\Lambda(n)}{\log n} = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right) + O\left(\frac{x}{\log^2 x}\right).$$

But recalling that $\psi(x) = \theta(x) + O(\sqrt{x}\log^2 x)$, we also have that

$$\sum_{2\leq n\leq x} \frac{\Lambda(n)}{\log n} = \frac{\psi(x)}{\log x} = \pi(x) + O(\sqrt{x}\log x).$$

Equating these two expressions, consolidating error terms, and isolating $\pi(x)$ gives the result that

$$\pi(x) = x/\log x + o(x/\log x).$$

For the other direction, we'll use partial summation on

$$\theta(x) = \sum_{n\leq x} a_n \log n,$$

where a_n is the prime indicator function. Doing this and using the hypothesis yields

$$\begin{aligned}\theta(x) = \sum_{n\leq x} a_n \log n &= \pi(x)\log x - \int_1^x \frac{\pi(t)}{t}dt\\ &= x + o(x) + O(\log x)\\ &= x + o(x).\end{aligned}$$

Again, we'll use the approximation $\theta(x) = \psi(x) + O(\sqrt{x}\log^2 x)$. Combining error terms yields $\psi(x) = x + o(x)$. $\square$

Fact (3.1.2)**.** *Suppose* $\lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)} = \alpha$. *We already proved that there exist* $c_1, c_2 \in \mathbb{R}_{>0}$ *such that* $\liminf_{x\to\infty} \frac{\pi(x)}{x/\log(x)} \geq c_1$ *and* $\limsup_{x\to\infty} \frac{\pi(x)}{x/\log(x)} \leq c_2$. *Then*

$\displaystyle\sum_{p\leq x}\frac{1}{p} = \alpha\log(\log(x)) + o(\log(\log(x)))$, *i.e. if* α *exists, then* $c_1 \leq \alpha \leq c_2$*. We showed that* $\displaystyle\sum_{p\leq x}\frac{1}{x^2}\ \ \log(\log(x))$.

Corollary. *If* $\displaystyle\lim_{x\to\infty}\frac{\pi(x)}{x/\log(x)} = \alpha$ *exists, then* $\alpha = 1$.

Proof. By partial summation with $a_n = \begin{cases} 1 & n \text{ prime} \\ 0 & \text{otherwise} \end{cases}$,

$$\sum_{p\leq x}\frac{1}{p} = \sum_{2\leq n\leq x}\frac{a_n}{n} = \frac{\pi(x)}{x} + \int_2^x \frac{\pi(t)}{t^2}dt =$$

$$O(\frac{1}{\log(x)}) + \int_2^y \frac{\pi(t)}{t^2}dt + \int_y^x \frac{\pi(t)}{t^2}dt =$$

$$O(\frac{1}{\log(x)} + y) + \int_x^y \frac{\alpha dt}{t\log(t)} + o(\int_x^y \frac{dt}{t\log(t)}) =$$

$$O(\frac{1}{\log(x)} + y) + \alpha\log(\log(x))) + O(\log(\log(y))) + o(\log(\log(x))) =$$

$$\alpha\log(\log(x)) + o(\log(\log(x)))$$

provided one can choose $y = y(x) \to \infty$ as $x \to \infty$ such that $y = o(\log(\log(x)))$. Taking $y = \dfrac{\log(\log(x))}{\log(\log(\log(x)))}$ works. □

Section 3.2: Nonvanishing of Dirichlet Series on Re(s) = 1

02/27/13

We could say that there are two ingredients to the proof of the prime number theorem:

1. The meromorphic continuation of $\zeta(s)$ to $\mathrm{Re}(s) = 1$.
2. The nonvanishing of $\zeta(s)$ on $\mathrm{Re}(s) = 1$.

The first of these we already have - in fact, we have more than we need. Recall in (2.1.6) we showed that

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dt,$$

and so equivalently

$$(s-1)\zeta(s) = s - s(s-1)\int_1^\infty \frac{\{x\}}{x^{s+1}}dt.$$

The right-hand side of this second equation is analytic for $\mathrm{Re}(s) > 0$, which means the expression for $\zeta(s)$ derived in (2.1.6) is meromorphic for $\mathrm{Re}(s) > 0$. Furthermore, we know that it has only one pole - a simple pole at $s = 1$ with residue 1.

Some work still needs to be done to obtain the second ingredient listed above. We've aluded to the following fact before when we concluded our discussion on infinite products, but we'll make it explicit here.

Fact (3.2.2)**.** *If $Re(s) > 1$, then $\zeta(s) \neq 0$.*

Proof. For $\mathrm{Re}(s) > 1$, we can use the Euler product to write

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} = \prod_p \left(1 + \frac{1}{p^s - 1}\right).$$

Therefore, to show that $\zeta(s)$ is nonzero, it suffices to show that

$$\left|\sum_p \frac{1}{p^s - 1}\right| < \infty.$$

Observe that after setting $s = \sigma + it$, we have

$$\begin{aligned} \left|\sum_p \frac{1}{p^s - 1}\right| &\leq \sum_p \left|\frac{1}{p^s - 1}\right| \\ &= \sum_p \frac{1}{p^\sigma + 1} \\ &\leq \sum_{n \geq 1} \frac{1}{n^\sigma + 1}, \end{aligned}$$

which converges for $\sigma > 1$. □

Fact (3.2.3)**.** *Let $s = \sigma + it$ and suppose $\sigma > 1$. Then*

$$\mathrm{Re}(\log \zeta(s)) = \sum_{n \geq 1} \frac{\Lambda(n)}{n^\sigma \log n} \cos(t \log n).$$

Proof. Observe that for $\sigma > 1$ we have

$$\begin{aligned} \log \zeta(s) &= -\sum_p \log(1 - p^{-s}) \\ &= \sum_p \sum_{k \geq 1} \frac{1}{kp^{ks}} \\ &= \sum_{k \geq 1} \frac{1}{k} \sum_p \frac{1}{p^{k\sigma}} p^{-ikt}. \end{aligned}$$

To combine these two summations, take $n = p^k$ and use the von Mangoldt function to identify the primes. That is, write

$$\begin{aligned}\log\zeta(s) &= \sum_{n\geq 1}\frac{\Lambda(n)}{n^\sigma\log n}n^{-it}\\ &= \sum_{n\geq 1}\frac{\Lambda(n)}{n^\sigma\log n}(\cos(t\log n)+i\sin(t\log n)),\end{aligned}$$

where from here we can pick out the real part. □

Fact (3.2.4)**.** *For $s = \sigma + it$ and $\sigma > 0$, we have*

$$\mathrm{Re}[3\log\zeta(\sigma)+4\log\zeta(\sigma+it)+\log\zeta(\sigma+2it)]\geq 0.$$

Proof. First, recall the double angle formula for cosine

$$\cos 2\theta = 2\cos^2\theta - 1.$$

Using the result from (3.2.3), write

$$\begin{aligned}\mathrm{Re}[3\log\zeta(\sigma)+4\log\zeta(\sigma+it)+\log\zeta(\sigma+2it)] &= \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^\sigma\log n}(3+4\cos(t\log n)+\cos(2t\log n))\\ &= \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^\sigma\log n}(3+4\cos(t\log n)+2\cos^2(t\log n)-1)\\ &= 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^\sigma\log n}(1+\cos(t\log n))^2,\end{aligned}$$

which is certainly greater than 0. □

Fact (3.2.5)**.** *For $s = \sigma + it$ and $\sigma > 0$, we have*

$$|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)|\geq 1.$$

Deduce from this that $\zeta(s)$ is nonzero on Re$(s) = 1$ (except when $s = 1$).

Proof. Taking the result from (3.2.4) and exponentiating both sides gives the first claim. To lead to a contradiction, suppose $\zeta(s)$ has a zero of order m at $s = 1 + it$ for some nonzero t. Set c to be the residue of this pole, that is

$$\lim_{\sigma\to 1^+}\frac{\zeta(\sigma+it)}{(\sigma-1)^m} = c \neq 0.$$

Taking the result from (3.2.5) and multiplying both sides by $(\sigma-1)^{3-4m}$ yields

$$|(\sigma-1)^3\zeta^3(\sigma)(\sigma-1)^{-4m}\zeta^4(\sigma+it)\zeta(\sigma+2it)|\geq(\sigma-1)^{3-4m},$$

where upon taking $\sigma \to 1^+$ we obtain

$$|1 \cdot c^4 \cdot \sigma(1+2it)| \geq \lim_{\sigma \to 1^+} (\sigma - 1)^{3-4m}.$$

If this is to be true, then the right-hand side cannot diverge to infinity. So we must have

$$3 - 4m \geq 0 \quad \Rightarrow \quad m = 0,$$

because m is an integer. Contradiction. Therefore, $\zeta(s)$ does not vanish on $\mathrm{Re}(s) = 1$ except when $s = 1$. □

1 March 2013

Last time:

$$\zeta(\sigma + it) \neq 0 \text{ if we have that } \sigma \leq 1, t \neq 0$$

Suppose $\chi \neq \chi^2$ ($\chi \neq \chi_0$ is not real)

We can use the same techniques as above to reprove the non-vanishing of L-series at $s = 1$ with nonreal nontrivial character (already proved in (2.3.10)).

Exercise. *Let χ be a non-real, non-trivial character* mod q *and write $s = \sigma + it$. If $\sigma > 1$, then*

$$\log L(\sigma, \chi) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma \log n} (\cos(t \log n - n\theta) - i \sin(t \log n - n\theta))$$

and

$$\log L(\sigma, \chi^2) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma \log n} (\cos(t \log n - 2n\theta) - i \sin(t \log n - 2n\theta))$$

Proof. From (2.3.2) we have the expression

$$\begin{aligned}
\log L(\sigma, \chi) &= \sum_{k \geq 1} \frac{1}{k} \sum_p \frac{\chi(p)^k}{p^{ks}} \\
&= \sum_{k \geq 1} \frac{1}{k} \sum_p \frac{\chi(p^k)}{p^{k\sigma}} p^{-itk}.
\end{aligned}$$

Setting $\chi(1) = e^{i\theta}$, taking $n = p^k$, and using the von Mangoldt function to identify the primes (just like in 3.2.3) yields

$$\begin{aligned}
\log L(\sigma, \chi) &= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma \log n} \chi(n) n^{-it} \\
&= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma \log n} e^{i(n\theta - t \log n)} \\
&= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma \log n} (\cos(t \log n - n\theta) - i \sin(t \log n - n\theta)).
\end{aligned}$$

Furthermore, if $\chi(1) = e^{i\theta}$, then $\chi^2(1) = e^{2i\theta}$ and so similarly

$$\log L(\sigma, \chi^2) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma \log n}(\cos(t \log n - 2n\theta) - i \sin(t \log n - 2n\theta)).$$

$\square$

Exercise. *For $s = \sigma + it$ with $\sigma > 0$, use the previous result to deduce that for a non-real non-trivial character χ,*

$$Re[3 \log \zeta(\sigma) + 4 \log L(\sigma, \chi) + \log L(\sigma, \chi^2)] \geq 0.$$

Proof. As before, we use the double angle formula

$$\cos 2x = 2 \cos^2 x - 1.$$

From the previous exercise, we have that

$$\begin{aligned} Re[3 \log \zeta(\sigma) + 4 \log L(\sigma, \chi) + \log L(\sigma, \chi^2)] &= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma \log n}(3 + 4 \cos(-n\theta) + \cos(-2n\theta)) \\ &= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma \log n}(3 + 4 \cos(-n\theta) + 2 \cos^2(-n\theta) - 1) \\ &= 2 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma \log n}(1 + \cos(-n\theta))^2, \end{aligned}$$

which is certainly greater than 0. $\square$

Corollary. *As an immediate corollary, we have that for a non-real, non-trivial character χ,*

$$|\zeta^3(\sigma) L^4(\sigma, \chi) L(\sigma, \chi^2)| \geq 1.$$

Proof. Exponentiate the result from the previous exercise. $\square$

Exercise. *Show that for non-real, non-trivial χ, $L(1, \chi)$ is nonzero.*

Proof. Suppose $L(1, \chi)$ has a zero of order m and residue c at some $t \neq 0$. That is,

$$\lim_{\sigma \to 1^+} \frac{L(\sigma, \chi)}{(\sigma - 1)^m} = c \neq 0.$$

Taking the result from the previous exercise and multiplying both sides by $(\sigma - 1)^{3-4m}$ yields

$$|(\sigma - 1)^3 \zeta^3(\sigma) L^4(\sigma, \chi)(\sigma - 1)^{-4m} L(\sigma, \chi^2)| \geq (\sigma - 1)^{3-4m}.$$

Upon taking $\sigma \to 1^+$ we obtain

$$|1 \cdot c^4 \cdot L(1, \chi^2)| \geq \lim_{\sigma \to 1^+} (\sigma - 1)^{3-4m}.$$

From the corollary to (2.3.4), $L(1, \chi^2)$ converges, and so the right-hand side must also converge. Therefore we have

$$3 - 4m \geq 0 \quad \Rightarrow \quad m = 0,$$

since m is an integer. Contradiction. So $L(1, \chi)$ is nonzero for non-real, non-trivial χ. □

Corollary. *$L(\sigma + it, \chi) \neq 0$ if $\sigma \leq 1$ and $t \in \mathbb{R}$ and $\chi \neq \chi^2$.*

Fact 3.2.6:

The function $\dfrac{-\zeta(s)}{\zeta(s)}$ has meromorphic continuity to $Re(s) \leq 1$ with only a simple pole at $s = 1$ with residue 1.

Proof.

Recall: $\zeta(s)$ and also $\zeta(s)$ have meromorphic continuity to $Re(s) > 0$ with poles only at $s = 1$.

Recall:

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \leq 1} \frac{\Lambda(n)}{n^s}$$

Also, $\zeta(s) \neq 0$ if $Re(s) \leq 1$.

$\dfrac{-\zeta(s)}{\zeta(s)}$ has meromorphic continuity to $Re(s) \leq 1$ with a pole only at $s = 1$ at order 1.

Recall (3.2.1)

$$(s-1)\zeta(s) = s(1 - (s-1)\int_1^\infty \frac{\{x\}}{x^{s+1}} dx$$

Note: $f(s)$ is analytic for $Re(s) > 0$.

$$\Rightarrow (s-1)\zeta(s) = sf(s) \;\text{ for } 1 \leq Re(s)$$

$$\Rightarrow \zeta(s) + (s-1)\zeta'(s) = f(s) + sf'(s)$$

$$\Rightarrow 1 + (s-1)\frac{\zeta'(s)}{\zeta(s)} = \frac{f(s)}{\zeta(s)} + \frac{sf'(s)}{\zeta(s)}$$

$$\Rightarrow (s-1)\frac{\zeta'(s)}{\zeta(s)} = -1 + \frac{f(s)}{\zeta(s)} + \frac{sf'(s)}{\zeta(s)}$$

$$\lim_{s\to 1+}[(s-1)\frac{\zeta'(s)}{\zeta(s)}] = -1$$

So $\dfrac{-\zeta'(s)}{\zeta(s)}$ has a simple pole at $s=1$ of residue 1.

$\square$

Exercise (3.2.7)**.** *Show that*

$$\frac{\sin{(n+\frac{1}{2})\theta}}{2\sin\frac{\theta}{2}} = \frac{1}{2} + \cos\theta + \cos 2\theta + \ldots + \cos n\theta.$$

Proof. Starting with the right-hand side, observe that this expression is the real part of the quantity

$$-\frac{1}{2} + (1 + e^{i\theta} + e^{i2\theta} + \ldots + e^{in\theta}),$$

which we can rewrite using a finite geometric sum as

$$-\frac{1}{2} + \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}. \tag{3}$$

Observe that multiplying the denominator by $e^{-i\theta/2}$ yields

$$e^{i\theta/2} - e^{-i\theta/2} = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2} - \cos\frac{\theta}{2} + i\sin\frac{\theta}{2} = 2i\sin\frac{\theta}{2},$$

so (1) can be equivalently expressed as

$$\begin{aligned}
-\frac{1}{2} + \frac{(e^{i(n+1)\theta} - 1)e^{-i\theta/2}}{2i\sin\frac{\theta}{2}} &= -\frac{1}{2} + \frac{e^{i(n+1/2)\theta} - e^{-i\theta/2}}{2i\sin\frac{\theta}{2}} \\
&= -\frac{1}{2} + \frac{\cos{(n+\frac{1}{2})\theta} + i\sin{(n+\frac{1}{2})\theta} - \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}{2i\sin\frac{\theta}{2}} \\
&= -\frac{1}{2} + \frac{i\cos{(n+\frac{1}{2})\theta} + \sin{(n+\frac{1}{2})\theta} - i\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}} \\
&= \frac{i\cos{(n+\frac{1}{2})\theta} + \sin{(n+\frac{1}{2})\theta} - i\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}.
\end{aligned}$$

At this point, all that remains is to pick out the real part of this quantity and confirm that this matches the left-hand side of the claim. $\square$

Ikehera Weiner Theorem

Review of Fourier Analysis

Definition. The Schwartz space of rapidly decreasing function is

$$S(\mathbb{R}) = \{f \in \mathbb{C}^\infty(\mathbb{R}) | \lim_{|x|\to\infty} x^n f^{(n)}(x) = 0, \forall m, n \in \mathbb{R}\}$$

Example: $f_m(t) = e^{-mt^2} \in S(\mathbb{R})(m > 0)$.

Definition. For $f \in S(\mathbb{R})$ we define the Fourier (inversion) of f by

$$\widehat{f}(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{itx}dt$$

Fact: $f(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \widehat{f}(t)e^{itx}$.

Proof. exercise

□

Corollary. *Note:* $\widehat{f}(x-y) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(t)e^{-itx+ity}dt.$

$\widehat{f}(x-y)$ *and* $f(t)e^{ity}$ *are Fourier inversions of each other.*

Parserol's Identity $\displaystyle\int_{-\infty}^{\infty} f(x)q(x)dx = \int_{-\infty}^{\infty} \widehat{f}(t)\widehat{g}(t)dt.$

Note: The formulas can be extended to $L^2(\mathbb{R})$.

Riemann-Lebesgue Lemma

$$\lim_{\lambda\to\infty} [\int_{-\infty}^{\infty} f(t)e^{ixt}dt] = 0$$

for every absolute convergent function f.

Fejer Kernel

$$K_\lambda(x) = \frac{\sin^2(\lambda x}{\lambda x^2}$$

where $K_\lambda = 2\sqrt{2\pi}(1 - \dfrac{|x|}{2\lambda}$ if $|x| \leq 2\lambda$, and 0 otherwise.

4 March 2013

Ikehara-Wiener Theorem
Let $F(s) = \sum_{n \geq 1} \frac{b_n}{n^s}$ be a Dirichlet series with $b_n \in \mathbb{R}_{>0}$ and absolute convergence for $\mathrm{Re}(s) \geq 1$. Suppose also that $F(s)$ can be meromorphically continued to $\mathrm{Re}(s) \geq 1$ with only a simple pole at $s = 1$ of residue $\mathrm{R} \geq 0$.
Then, $B(x) = \sum_{n \leq x} b_n = \mathrm{R}x + o(x)$ as $x \to \infty$.

Proof. Without loss of generality, we may assume $\mathrm{R} > 0$ because if $\mathrm{R}=0$, $F(s)$ is analytic on $\mathrm{Re}(s) \geq 1$ and we may replace $F(s)$ by $F(s) + \zeta(s)$ where $F(s) = \sum_{n \geq 1} \frac{1 + b_n}{n^s}$ and this satisfies the hypothesis with $\mathrm{R}=1$. Applying the theorem, then yields $\sum_{n \leq x} \frac{1 + b_n}{n^s} = x + o(x) \implies \sum_{n \leq x} b_n = o(x)$. If $\mathrm{R} \neq 0, 1$, then replace $F(s)$ by $\sum_{n \leq x} \frac{\frac{b_n}{\mathrm{R}}}{n^s} = G(s)$. Then, G satisfies the theorem by $\mathrm{R}=1$ and the theorem gives $\frac{1}{\mathrm{R}} \sum_{n \leq x} b_n = x + o(x) \implies \sum_{n \leq x} b_n = \mathrm{R}x + o(x)$.
So, we assume $R = 1$. Then, by partial summation, letting $f(t) = \frac{1}{t^s}$,

$$F(s) = \lim_{N \to \infty} \left(\sum_{n=1}^{N} \frac{b_n}{n^s} \right) \lim_{N \to \infty} \left(\frac{B(N)}{N^s} + s \int_1^N \frac{B(t)}{t^{s+1}} \mathrm{d}t \right)$$

Since $F(s)$ is analytic for $\mathrm{Re}(s) > 1$, $\frac{B(N)}{N^s} = O(1)$ as $N \to \infty$ for $\mathrm{Re}(s) > 1$.
In other words, $B(N) = O(N^{\mathrm{Re}(s) > 1})$ for any s for $\mathrm{Re}(s) > 1$.
That is, $B(N) = O(N^{1+\delta})$ for all $\delta > 0$.
So, for any s with $\mathrm{Re}(s) > 1 + \delta > 0$, $\frac{B(N)}{N^s} \to 0$ as $N \to \infty$.
So, for $\mathrm{Re}(s) > 1, F(s) = s \int_1^\infty \frac{B(t)}{t^{s+1}} \mathrm{d}t = s \int_1^\infty \frac{B(t)}{t^s} \frac{\mathrm{d}}{t^s}$. Setting $t = e^u \implies u = \ln(t)$ so $\mathrm{d}t = e^u \mathrm{d}u \implies \mathrm{d}u = \frac{\mathrm{d}t}{t}$,

$$F(s) = \int_0^\infty \frac{B(e^u)}{e^{su}} \mathrm{d}u \quad \frac{F(s)}{s} = \int_0^\infty B(e^u)\, e^{-su} \mathrm{d}u$$

Note:

$$\int_0^\infty e^{-u(s-1)} \mathrm{d}u = \frac{-1}{s-1} e^{-u(s-1)} \Big|_{u=0}^{\infty} = \frac{1}{s-1}$$

Write $s = 1 + \delta + ix$ where $\delta > 0$. Then,

$$\begin{aligned}\frac{F(s)}{s} &= \frac{F(1+\delta+ix)}{1+\delta+ix} \\ &= \int_0^\infty B\left(e^u\right) e^{-u(1+\delta+ix)} \mathrm{d}u\end{aligned}$$

$$\int_0^\infty B\left(e^u\right) e^{-u} e^{-u\delta} e^{-uxi} \mathrm{d}u$$

So,

$$\frac{F(s)}{s} - \frac{1}{s-1} = \int_0^\infty \left[B\left(e^u\right) e^{-u} - 1\right] e^{-u\delta} e^{-iux} \mathrm{d}u.$$

Let $g(u) = B\left(e^u\right) e^{-u}$

$h_\delta(x) = \dfrac{F(1+\delta+ix)}{1+\delta+ix} - \dfrac{1}{\delta+ix} \implies h(x) = \dfrac{F(1+ix)}{1+ix} - \dfrac{1}{1+ix}$. Where $h(t)$ just has $\delta = 0$.

We have that $\dfrac{1}{\delta+ix}$ is analytic at $s = 1$ and $\dfrac{1}{1+ix}$ is well defined for all $x \in \mathbb{R}$ as well as continuous and infinitely differentiable.

Aside: we want $g(u) \to 1$ as $u \to \infty$.

Exercise

Both of these functions are square integrable because $B(x) \ll \sum_{n\geq 1} b_n \left(\frac{x}{n}\right)^c$ for all $c > 1$.

Continuing with the proof, then, notice that $\left[\sqrt{2\pi}(\widehat{g(u)-1})e^{u\delta}\right](t) = h_\delta(t)$ so that, using Parseval's formula, we get

$$\sqrt{2\pi}\int_{-\infty}^\infty (g(u)-1)e^{-u\delta}k_\lambda(u)\mathrm{d}u = \int_{-\infty}^\infty h_\delta(x)\widehat{k_\lambda(x)}\mathrm{d}x.$$

Also, using a property of translation with Fourier transform,

$$\sqrt{2\pi}\int_{-\infty}^\infty (g(u)-1)e^{-u\delta}k_\lambda(u-v)\mathrm{d}u = \int_{-\infty}^\infty h_\delta(x)\widehat{k_\lambda(x)}e^{-ixv}\mathrm{d}x.$$

Recall: Since $\widehat{k_\lambda(x)}$ has compact support $[-2\lambda, 2\lambda]$, the limits as $\delta \to 0$ of the right hand sides (of our above equations) exist, and thus the same is true for the left hand side(s). Thus,

$$\sqrt{2\pi}\int_{-\infty}^\infty (g(u)-1)k_\lambda(u-v)\mathrm{d}u = \int_{-\infty}^\infty h(x)\widehat{k_\lambda(x)}e^{-ixv}\mathrm{d}x.$$

And by the Riemann-Lebesgue Lemma, we have that the RHS $\to 0$ as $v \to \infty$.

After moving the constant to the other side, this argument leads to the following statement about the left-hand side:

$$\lim_{\delta\to 0}\lim_{v\to\infty}\int_{-\infty}^\infty (g(u)-1)k_\lambda(u-v)du = 0. \tag{4}$$

Exercise. *Show that*

$$\int_{-\infty}^{\infty} K_\lambda(x)dx = \int_{-\infty}^{\infty} \frac{\sin^2 \lambda x}{\lambda x^2}dx = \pi.$$

Proof. Set

$$f(t) = \begin{cases} 1 & \text{if } |t| \le \lambda \\ 0 & \text{otherwise.} \end{cases}$$

Of interest is the Fourier transform of f:

$$\begin{aligned}
\widehat{f}(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-itx} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\lambda}^{\lambda} e^{-itx} dx \\
&= \frac{1}{\sqrt{2\pi}} \left(-\frac{1}{it} e^{-itx} \Big|_{-\lambda}^{\lambda} \right) \\
&= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{it} \left(e^{it\lambda} - e^{-it\lambda} \right) \\
&= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{it} (2i \sin t\lambda) \\
&= \frac{2 \sin t\lambda}{t\sqrt{2\pi}}.
\end{aligned}$$

On one side of Parseval's identity, we have that

$$\int_{-\infty}^{\infty} f^2(t)dt = \int_{-\lambda}^{\lambda} 1 dt = 2\lambda$$

so this means that

$$2\lambda = \int_{-\infty}^{\infty} \hat{f}^2(t)dt = \int_{-\infty}^{\infty} \frac{2\sin^2 t\lambda}{t^2 \pi} dt.$$

Rearranging this final equation gives the desired result. □

Using this result, rewrite (1) as

$$\lim_{v\to\infty} \int_{-\infty}^{\infty} g(u) k_\lambda(u-v) du = \pi, \tag{5}$$

where the condition that $\delta \to 0$ will remain unstated for the remainder of the proof. Consider the change of variables $u = v + \alpha/\lambda$ so that $du = d\alpha/\lambda$. Rewriting (2) in this fashion we see that

$$\begin{aligned}
\pi &= \lim_{v\to\infty} \int_{-\infty}^{\infty} g\left(v + \frac{\alpha}{\lambda}\right) k_\lambda \left(\frac{\alpha}{\lambda}\right) \frac{d\alpha}{\lambda} \\
&= \lim_{v\to\infty} \int_{-\infty}^{\infty} g\left(v + \frac{\alpha}{\lambda}\right) \frac{\sin^2 \alpha}{\alpha^2} d\alpha.
\end{aligned} \tag{6}$$

We now begin building an upper bound. Recall that $B(x)$ is monotonic increasing, so for $u_2 \geq u_1$ we have that $B(e^{u_2}) \geq B(e^{u_1})$ and in particular, using our function $g(u) = B(e^u)e^{-u}$ that

$$g(u_2) \geq g(u_1)e^{u_1 - u_2}.$$

So for $|\alpha| \leq \sqrt{\lambda}$, certainly $\frac{\alpha}{\lambda} \geq -\frac{1}{\sqrt{\lambda}}$ and thus

$$\begin{aligned} g\left(v + \frac{\alpha}{\lambda}\right) &\geq g\left(v - \frac{1}{\sqrt{\lambda}}\right) e^{\frac{-1}{\sqrt{\lambda}} + \frac{\alpha}{\lambda}} \\ &\geq g\left(v - \frac{1}{\sqrt{\lambda}}\right) e^{\frac{-2}{\sqrt{\lambda}}}. \end{aligned}$$

Returning to (3), we now see the power of the most recent change in variables. We now have a bound on part of the integrand that does not depend on α. In particular, write

$$\begin{aligned} \pi &= \lim_{v\to\infty} \int_{-\infty}^{\infty} g\left(v + \frac{\alpha}{\lambda}\right) \frac{\sin^2\alpha}{\alpha^2} d\alpha \\ &\geq \lim_{v\to\infty} \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} g\left(v + \frac{\alpha}{\lambda}\right) \frac{\sin^2\alpha}{\alpha^2} d\alpha \\ &\geq \lim_{v\to\infty} \sup g\left(v - \frac{1}{\sqrt{\lambda}}\right) e^{\frac{-2}{\sqrt{\lambda}}} \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \frac{\sin^2\alpha}{\alpha^2} d\alpha. \end{aligned}$$

Upon setting $w = v - \dfrac{1}{\sqrt{\lambda}}$ we arrive at

$$\lim_{w\to\infty} \sup g(w) \leq \frac{\pi e^{\frac{2}{\sqrt{\lambda}}}}{\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \frac{\sin^2\alpha}{\alpha^2} d\alpha}. \tag{7}$$

To simplify the integral in this expression, observe that the integrand is even and so

$$\begin{aligned} \pi = \int_{-\infty}^{\infty} \frac{\sin^2\alpha}{\alpha^2} d\alpha &= \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \frac{\sin^2\alpha}{\alpha^2} d\alpha + 2\int_{\sqrt{\lambda}}^{\infty} \frac{\sin^2\alpha}{\alpha^2} d\alpha \\ &\leq \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \frac{\sin^2\alpha}{\alpha^2} d\alpha + 2\int_{\sqrt{\lambda}}^{\infty} \frac{1}{\alpha^2} d\alpha \\ &= \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \frac{\sin^2\alpha}{\alpha^2} d\alpha + O\left(\frac{1}{\sqrt{\lambda}}\right). \end{aligned}$$

Therefore, rewriting (4) we have

$$\lim_{w\to\infty} \sup g(w) \leq \frac{\pi e^{\frac{2}{\sqrt{\lambda}}}}{\pi + O\left(\frac{1}{\sqrt{\lambda}}\right)}$$

and driving the arbitrary $\lambda \to \infty$ yields

$$\lim_{w\to\infty} \sup g(w) \le 1.$$

It remains to be shown that the limit of the infimums is bounded below by 1, but the strategy is very similar to the one used for the supremums. In particular, recall that for $u_2 \ge u_1$ we had that $g(u_2) \ge g(u_1)e^{u_1-u_2}$, but now we're interested in the equivalent statement

$$g(u_1) \le g(u_2)e^{u_2-u_1}.$$

Now, for $|\alpha| \le \sqrt{\lambda}$ we have that $\frac{\alpha}{\lambda} \le \frac{1}{\sqrt{\lambda}}$ and thus

$$\begin{aligned} g\left(v+\frac{\alpha}{\lambda}\right) &\le g\left(v+\frac{1}{\sqrt{\lambda}}\right)e^{\frac{1}{\sqrt{\lambda}}-\frac{\alpha}{\sqrt{\lambda}}} \\ &\le g\left(v+\frac{1}{\sqrt{\lambda}}\right)e^{\frac{2}{\sqrt{\lambda}}}. \end{aligned}$$

Returning again to (3), use this new bound to write

$$\begin{aligned} \pi &= \lim_{v\to\infty} \int_{-\infty}^{\infty} g\left(v+\frac{\alpha}{\lambda}\right)\frac{\sin^2\alpha}{\alpha^2}d\alpha \\ &\le \lim_{v\to\infty} \inf g\left(v+\frac{1}{\sqrt{\lambda}}\right)e^{\frac{2}{\sqrt{\lambda}}}\int_{-\infty}^{\infty}\frac{\sin^2\alpha}{\alpha^2}d\alpha \end{aligned}$$

or more appropriately

$$\pi + O\left(\frac{1}{\sqrt{\lambda}}\right) \le \lim_{v\to\infty} \inf g\left(v+\frac{1}{\sqrt{\lambda}}\right)e^{\frac{2}{\sqrt{\lambda}}}\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}}\frac{\sin^2\alpha}{\alpha^2}d\alpha$$

so that setting $w = v + \frac{1}{\sqrt{\lambda}}$, rearranging, and then driving $\lambda \to \infty$ we discover that

$$\lim_{w\to\infty} \inf g(w) \ge \frac{\left(\pi + O\left(\frac{1}{\sqrt{\lambda}}\right)\right)e^{-\frac{2}{\sqrt{\lambda}}}}{\int_{-\infty}^{\infty}\frac{\sin^2\alpha}{\alpha^2}d\alpha} = \frac{\left(\pi + O\left(\frac{1}{\sqrt{\lambda}}\right)\right)e^{-\frac{2}{\sqrt{\lambda}}}}{\left(\pi + O\left(\frac{1}{\sqrt{\lambda}}\right)\right)} = 1.$$

This completes the proof. We've shown now that $\lim_{u\to\infty} g(u) = 1$ and so $\lim_{t\to\infty} B(t) = t$. Equivalently,

$$B(t) = t + o(t).$$

□

Corollary (Prime Number Theorem)**.** *As an immediate corollary,*

$$\pi(x) \sim x/\log x$$

.

Proof. The function $-\zeta'(s)/\zeta(s) = D(\Lambda, s)$ satisfies the conditions of the Ikehara-Wiener Theorem with residue 1, so we have that

$$D(\Lambda, s) = x + o(x).$$

In particular then,

$$\lim_{x\to\infty} \psi(x) = \lim_{x\to\infty} \sum_{n\leq x} \Lambda(n) = x$$

and so by (3.1.11) we conclude that $\pi(x) \sim x/\log x$. □

The next couple results extend the IWT to other contexts. In particular, what if we have some negative coefficients in our sequence, or what if our sequence consists of complex numbers?

Fact (3.3.3)**.** *Suppose $f(s) = \sum_{n=1}^{\infty} a_n/n^s$, with real a_n and $f(s)$ absolutely convergent for* $\mathrm{Re}(s) > 1$*. If $f(s)$ has meromorphic continuation to* $\mathrm{Re}(s) \geq 1$ *with at worst a simple pole at $s = 1$ of residue R_f, and if there is some real sequence b_n with $|a_n| \leq b_n$, where the function $F(s) = \sum_{n=1}^{\infty} b_n/n^s$ satisfies the hypotheses of the IWT with residue R_F, then the IWT holds for $f(s)$. That is*

$$A(x) := \sum_{n\leq x} a_n = R_f x + o(x).$$

Proof. Consider the function $G(s) := F(s) - f(s)$, which has positive real coefficients. $G(s)$ satisfies the hypotheses of the IWT with residue $R_F - R_f$ and so we have that

$$\sum_{n\leq x}(b_n - a_n) = (R_F - R_f)x + o(x).$$

By assumption, $F(s)$ also satisfies the conditions of the IWT with residue R_F, so we also have that

$$\sum_{n\leq x} b_n = R_F x + o(x).$$

Subtracting the first equation from the second gives the result. □

Fact (3.3.4)**.** *The previous fact also holds if a_n is a complex sequence.*

13 March 2013

Fact: Let $q \in \mathbb{N}$ and define $\psi(\chi, q, a) = \sum_{n<\chi, n\equiv a \pmod q} \Lambda(n)$.

Then $\psi(\chi, q, a) \sim \dfrac{\chi}{\varphi(q)}$ provided $(a, q) = 1$.

Proof.

Consider $f(s) = \dfrac{1}{\varphi(q)} \sum_{\chi\in\widehat{\mathbb{Z}/q\mathbb{Z}}} \overline{\chi}(a)(\dfrac{-L'(s,\chi)}{L(s,\chi)})$.

Observations:

- $L(s,\chi) = \prod_p (1 - \frac{\chi(p)}{p^s})^{-1}$.

$$\Rightarrow \log L(s,\chi) = \sum_p \log(1 - \frac{\chi(p)}{p^s}) = \sum_p \sum_{k=1}^{\infty} \frac{\chi(p)^k}{kp^{ks}}$$

$$= \sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{\log(n) n^s}$$

- $$\Rightarrow \frac{L'(s,\chi)}{L(s,\chi)} = \frac{d}{ds}\log(L(s,\chi)) = -\sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{n^s}$$

- $$f(s) = \frac{-1}{\varphi(a)} \sum_{\chi \pmod q} \overline{\chi}(a) \sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{n^s}$$

$$= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \chi(na^{-1})$$

$$= \sum_{n=1, n \equiv a \pmod q}^{\infty} \frac{\Lambda(n)}{n^s}$$

- Recall $L(s,\chi)$ is analytic for $Re(s) > 0$ if $\chi \neq \chi_0$, and if $\chi = \chi_0$, $L(s,\chi_0)$ is analytic for $Re(s) > 1$ and has meromorphic continuation to $Re(s) > 0$ with a simple pole at $s = 1$.

$$L(s,\chi_0) = \prod_p (1 - \frac{\chi_0(p)}{p^s})^{-1} = \prod_{p \nmid q} (1 - \frac{1}{p^s})^{-1}$$

$$= \zeta(s) \prod_{p|q} (1 - \frac{1}{p^s})$$

$$Re(L(s,\chi), 1) = \frac{\varphi(q)}{q}$$

Also $L(s,\chi) \neq 0$ for $Re(s) \geq 1$.

So $f(s)$ has meromorphic continuation to $Re(s) \geq 1$ with a simple pole at $s = 1$.

Recall: $h(s) = (s-1)L(s,\chi_0)$ is analytic for $Re(s) > 0$.

$$\Rightarrow \frac{h'(s)}{h(s)} = \frac{1}{s-1} + \frac{L'(s,\chi_0)}{L(s, chi_0)}$$

$$\Rightarrow \frac{-L'(s,\chi_0)}{L(s,\chi_0)} = \frac{1}{s-1} - \frac{h'(s)}{h(s)}$$

So, $Re_{s=1}(\frac{-L'(s,\chi)}{L(s,\chi)}) = 1$.

$$\Rightarrow Re_{s=1}(f(s)) = Re_{s=1}(\frac{1}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi}(a)(\frac{-L'(s,\chi)}{L(s,\chi)}))$$

$$= Re_{s=1}(\frac{-L'(s,\chi_0)}{\varphi(q)L(s,chi_0)}) = \frac{1}{\varphi(q)}$$

Applying I.W.T $\sum_{n\leq\chi, n\equiv q \pmod q} \Lambda(n) = \frac{\chi}{\varphi(q)} + o(\chi)$.

$$\sum_{n\equiv a \pmod q} frac\Lambda(n)\frac{\chi}{\varphi(q)} \to 1 \Leftrightarrow \prod(\chi, q, a) \sim \frac{\chi}{\varphi(q)\log x)}$$

□

Exercise. *As an immediately corollary, use this result to prove Dirichlet's theorem. That is, show that*

$$\pi(x,q,a) = \sum_{\substack{p\leq x\\ p\equiv a(q)}} 1 \sim \frac{x}{\phi(q)\log x}.$$

Proof. Supposing $\psi(x,q,a) \sim x/\phi(q)$, consider the sum

$$\sum_{\substack{n\leq x\\ n\equiv a(q)}} \frac{\Lambda(n)}{\log n}.$$

Taking $f(t) = 1/\log t$ and using partial summation yields

$$\sum_{\substack{n\leq x\\ n\equiv a(q)}} \frac{\Lambda(n)}{\log n} = \frac{\psi(x,q,a)}{\log x} + \int_2^x \frac{\psi(x,q,a)}{t\log^2 t} dt$$

$$= \frac{x}{\phi(q)\log x} + o\left(\frac{x}{\phi(q)\log x}\right) + O\left(\int_2^x \frac{dt}{t\log^2 t}\right),$$

whereby using the approximation of this integeral computed in (3.1.11) allows us to write

$$= \frac{x}{\phi(q)\log x} + o\left(\frac{x}{\phi(q)\log x}\right) + O\left(\frac{x}{\log^2 x}\right)$$
$$= \frac{x}{\phi(q)\log x} + o\left(\frac{x}{\phi(q)\log x}\right).$$

Seeing as this sum is precisely $\pi(x,q,a)$, we have our result. □

The next two results generalize the Ikehara-Wiener theorem to functions that have meromorphic continuation to any positive real number c.

Exercise (3.3.6)**.** *Suppose $F(s) = \sum_{n\geq 1} b_n/n^s$ is a Dirichlet series with positive real coefficients, which is convergent for Re$(s) > c$ for some real number c. Suppose further that $F(s)$ has meromorphic continuation to Re$(s) \geq c$, with at worst a simple pole at $s = c$ of residue R. Then*

$$B(x) := \sum_{n\leq x} b_n = \frac{R}{c}x^c + o(x^c).$$

Proof. Consider the function

$$G(s) = F(s+c-1) = \sum_{n\geq 1} \frac{b_n}{n^{c-1}} n^{-s},$$

which shifts F along the real line so that G satisfies now the conditions for the Ikehara-Wiener conjecture with residue R. That is,

$$D(x) := \sum_{n\leq x} \frac{b_n}{n^{c-1}} = Rx + o(x).$$

Now observe that we can write

$$B(x) = \sum_{n\leq x} \frac{b_n}{n^{c-1}} n^{c-1},$$

and so taking $f(t) = t^{c-1}$ and using partial summation we obtain

$$\begin{aligned} B(x) &= D(x)x^{c-1} - \int_1^x (c-1)D(t)t^{c-2}dt \\ &= Rx^c + o(x^c) - (c-1)\int_1^x Rt^{c-1}dt + o\left(\int_1^x (c-1)t^{c-1}dt\right) \\ &= Rx^c - (c-1)\frac{Rx^c}{c} + o(x^c) \\ &= \frac{Rx^c}{c} + o(x^c). \end{aligned}$$

□

Exercise (3.3.7)**.** *Suppose $f(s) = \sum_{n=1}^{\infty} a_n/n^s$ is a Dirichlet series with complex coefficients and is absolutely convergent for Re$(s) > c$. If $f(s)$ has meromorphic continuation to Re$(s) \geq c$ with at worst a simple pole at $s = c$ of residue R, and furthermore, if there is a function $g(s) = \sum_{n=1}^{\infty} b_n/n^s$ that satisfies the hypotheses in (3.3.6) with the added property that $|a_n| \leq b_n$, then*

$$A(x) := \sum_{n\leq x} a_n = \frac{R}{c}x^c + o(x^c).$$

Exercise (3.3.9)**.** *Let $c_n \geq 0$ and suppose*

$$c(x) := \sum_{n \leq x} c_n = Ax + o(x).$$

Use partial summation to show that

$$\sum_{n \leq x} \frac{c_n}{n} = A \log x + o(\log x).$$

Proof. Take $f(t) = 1/t$ and use partial summation to yield

$$\begin{aligned}
\sum_{n \leq x} \frac{c_n}{n} &= \frac{c(x)}{x} + \int_1^x \frac{c(t)}{t^2} dt \\
&= A + \int_1^x \frac{A}{t} dt + o\left(\int_1^x \frac{dt}{t} dt \right) \\
&= A + A \log x + o(\log x) \\
&= A \log x + o(\log x).
\end{aligned}$$

□

Section 4.1: Basic Integrals

25 March 2013

Idea
We are going to relate the behavior of $\sum_{n \leq x} a_n$ as $n \to \infty$ to $f(s) = \sum_{n \leq 1} \frac{a_n}{n^s}$.

Typically, we will compute $\oint_{\mathrm{Re}(s)=c} f(s) \mathrm{d}s = \lim_{N \to \infty} \int_{-N}^{N} f(c+it) \mathrm{d}t$.
We will now introduce some notation for ease: $\mathrm{Re}(s) = c$ will be written as (c).

Tool
Cauchy's Theorem relating the value of a line integral to the residue of the poles of the function (if any).

Section 4.1: Basic Integrals
Definition
Define

$$\delta(x) = \begin{cases} 0 & \text{if } 0 < x < 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 1 & \text{if } x > 1. \end{cases}$$

Fact

$$\delta(x) = \frac{1}{2\pi i} \oint_{(c)} \frac{x^s}{s} \mathrm{d}s.$$

Theorem 4.1.4
Put $I_c(x;R) = \dfrac{1}{2\pi i}\displaystyle\int_{c-iR}^{c+iR} \frac{x^s}{s}\mathrm{d}s$.
Notice that as $R \to \infty$, this is the same as $\mathrm{Re}(s) = c$.
Then, for $x > 0, c > 0, R > 0$, we have that

$$|I_c(x;R) - \delta(x)| < \begin{cases} x^c \min\left\{1, \dfrac{1}{R|\log x|}\right\} & \text{if } x \neq 1 \\ \dfrac{c}{R} & \text{if } x = 1. \end{cases}$$

Proof. <u>Case 1: $0 < x < 1$</u>
Consider the contour, K_u for $u > c$. Since $\dfrac{x^s}{s}$ has only a simple pole at $s = 0$,

$$\frac{1}{2\pi i}\oint_{K_u} \frac{x^s}{s}\mathrm{d}s = 0 = \delta(x)$$

so that

$$\begin{aligned}
&|\delta(x) - I_c(x;R)| = \\
&\qquad = \frac{1}{2\pi i}\left[\oint_{c+iR}^{u+iR}\frac{x^s}{s}\mathrm{d}s - \oint_{u-iR}^{u+iR}\frac{x^s}{s}\mathrm{d}s - \oint_{c-iR}^{u-iR}\frac{x^s}{s}\mathrm{d}s\right] \\
&\qquad \leq \left|\frac{1}{2\pi i}\oint_{c+iR}^{u+iR}\frac{x^s}{s}\mathrm{d}s\right| + \left|\frac{1}{2\pi i}\oint_{u-iR}^{u+iR}\frac{x^s}{s}\mathrm{d}s\right| + \left|\frac{1}{2\pi i}\oint_{c-iR}^{u-iR}\frac{x^s}{s}\mathrm{d}s\right|
\end{aligned}$$

Now, our aim is to estimate these terms.
We have

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int_{c+iR}^{u+iR}\frac{x^s}{s}\mathrm{d}s\right| &\overset{s=w+iR}{=} \left|\frac{1}{2\pi i}\int_c^u \frac{x^{w+iR}}{w+iR}\mathrm{d}w\right| \\
&\leq \frac{1}{2\pi}\int_c^u \frac{x^w}{|w+iR|}\mathrm{d}w \\
&\leq \frac{1}{2\pi}\int_c^u \frac{x^w}{R}\mathrm{d}w \\
&= \frac{1}{2\pi R}\int_c^u exp(|\log(x)|w)\mathrm{d}w \\
&= \frac{1}{2\pi R}\left|\left[\frac{1}{|\log(x)|}(x^u - x^c)\right]\right| \\
&\leq \frac{x^c}{2\pi R|\log(x)|}.
\end{aligned}$$

Similarly, we can show that $\left|\dfrac{1}{2\pi i}\displaystyle\int_{c-iR}^{u-iR}\frac{x^s}{s}\mathrm{d}s\right| \leq \dfrac{x^c}{2\pi R|\log(x)|}$.

Now, let's take care of the third integral. In particular, we have this estimate:

$$
\begin{aligned}
\left|\frac{1}{2\pi i}\int_{u-iR}^{u+iR}\frac{x^s}{s}\mathrm{d}s\right| &\overset{s=w+iR}{=} \left|\frac{i}{2\pi i}\int_{-R}^{R}\frac{x^{u+iw}}{u+iw}\mathrm{d}w\right| \\
&< \frac{1}{2\pi}\int_{-R}^{R}\frac{x^u}{|u|}\mathrm{d}w \\
&= \frac{x^u 2R}{2\pi|u|} \\
&= \frac{x^u R}{\pi|u|} \to 0 \text{ as } \; u \to \infty \;\text{ b/c } \; 0 < x < 1.
\end{aligned}
$$

So, we can select u sufficiently large so that the integral becomes as small as required, i.e.

$$\left|\frac{1}{2\pi i}\int_{c-iR}^{c+iR}\frac{x^s}{s}\mathrm{d}s\right| \leq \frac{x^c}{R\log(x)}.$$

So, we have for this case:

$$|I_c(x;R) - \delta(x)| = |I_c(x;R)| \leq \frac{x^c}{R\log(x)}.$$

All that remains to be shown is that the integral is bounded by x^c.
Consider the circle centered at 0 with radius $\sqrt{c^2+R^2}$. Let ζ_R be the arc running from $c-iR$ to $c+iR$ (Notice this is the same as I). So, we can bound our expression:

$$
\begin{aligned}
\left|\frac{1}{2\pi i}\int_{c-iR}^{c+iR}\frac{x^s}{s}\mathrm{d}s\right| &= \left|\frac{1}{2\pi i}\int_{\zeta_R}\frac{x^s}{s}\mathrm{d}s\right| \\
&\leq \frac{1}{2\pi}\int_{\zeta_R}\frac{x^c}{\sqrt{c^2+R^2}}\mathrm{d}s \\
&= \frac{x^c}{2\pi\sqrt{c^2+R^2}}\int_{\zeta_R}\mathrm{d}s \\
&= \frac{x^c}{2\pi\sqrt{c^2+R^2}}(\theta\sqrt{c^2+R^2}) \\
&\leq \frac{x^c}{2} \\
&< x^c.
\end{aligned}
$$

□

Now suppose $x = 1$. Using the substitution $s = c + it$, $ds = idt$, we compute

$$\begin{aligned}
I_c(1, R) &= \frac{1}{2\pi i} \int_{c-iR}^{c+iR} \frac{ds}{s} \\
&= \frac{1}{2\pi} \int_{-R}^{R} \frac{dt}{c + it} \\
&= \frac{1}{2\pi} \int_{-R}^{R} \frac{c - it}{c^2 + t^2} dt \\
&= \frac{1}{2\pi} \int_{-R}^{R} \frac{c}{c^2 + t^2} dt - \frac{i}{2\pi} \int_{-R}^{R} \frac{t}{c^2 + t^2} dt.
\end{aligned}$$

Notice that the first integrand is even and the second integrand is odd, which makes the remaining calculation much easier. Now write

$$\begin{aligned}
I_c(1, R) &= \frac{c}{\pi} \int_{0}^{R} \frac{dt}{c^2 + t^2} \\
&= \frac{1}{c\pi} \int_{0}^{R} \frac{dt}{1 + (t/c)^2} \\
&= \frac{1}{\pi} \arctan(t/c) \Big|_{t=0}^{t=R} \\
&= \frac{1}{\pi} \arctan(R/c).
\end{aligned}$$

In particular, as $R \to \infty$, we have that $I_c(1, R) \to \frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2} = \delta(1)$. Therefore, we can write

$$\begin{aligned}
|I_c(1, R) - \delta(1)| &= \frac{1}{c\pi} \int_{R}^{\infty} \frac{dt}{1 + (t/c)^2} \\
&\leq \frac{1}{c\pi} \int_{R}^{\infty} \frac{dt}{(t/c)^2} \\
&= \frac{c}{\pi t} \Big|_{t=R}^{t=\infty} \\
&< c/R.
\end{aligned}$$

Finally, suppose $x > 1$. This case proceeds almost identically to the first case. For $u > c$, consider the rectangular contour S_u with diagonal vertices $-u - iR$ and $c + iR$ and a counterclockwise orientation. By Cauchy's theorem,

$$\frac{1}{2\pi i} \oint_{S_u} \frac{x^s}{s} ds = 1 = \delta(x).$$

Recognizing that $I_c(x, R)$ is the right vertical dimension of this contour, we can use the triangle inequality to write

$$|I_c(x, R) - \delta(x)| \leq \frac{1}{2\pi} \left| \int_{-u-iR}^{c+iR} \frac{x^s}{s} ds \right| + \frac{1}{2\pi} \left| \int_{c-iR}^{-u-iR} \frac{x^s}{s} ds \right| + \frac{1}{2\pi} \left| \int_{-u-iR}^{-u+iR} \frac{x^s}{s} ds \right|.$$

As before, we bound these individual pieces. For the first, set $s = w + iR$ and write

$$\begin{aligned}
\frac{1}{2\pi}\left|\int_{-u+iR}^{c+iR} \frac{x^s}{s} ds\right| &= \frac{1}{2\pi}\left|\int_{-u}^{c} \frac{x^{w+iR}}{w+iR} dw\right| \\
&\leq \frac{1}{2\pi}\int_{-u}^{c} \frac{x^w}{R} dw \\
&= \frac{1}{2\pi R} \cdot \frac{x^w}{\log x}\Big|_{w=-u}^{w=c} \\
&\leq \frac{x^c}{2\pi R \log x} \quad \text{(because } x > 1\text{)}.
\end{aligned}$$

Similarly, the other horizontal piece follows the same bound:

$$\frac{1}{2\pi}\left|\int_{c-iR}^{-u-iR} \frac{x^s}{s} ds\right| \leq \frac{x^c}{2\pi R \log x}.$$

For the final piece, observe that upon setting $s = -u + it$, $ds = idt$, we discover

$$\begin{aligned}
\frac{1}{2\pi}\left|\int_{-u-iR}^{-u+iR} \frac{x^s}{s} ds\right| &= \frac{1}{2\pi}\left|\int_{-R}^{R} \frac{x^{-u+it}}{-u+it} dt\right| \\
&\leq \frac{1}{2\pi}\int_{-R}^{R} \frac{x^{-u}}{u} dt \\
&= \frac{x^{-u}R}{\pi u},
\end{aligned}$$

so this term can be made arbitrarily small for an appropriate choice of u. Therefore, we have the first result in this case, namely,

$$|I_c(x, R) - \delta(x)| < \frac{x^c}{R \log x}.$$

Now consider the same origin-centered circle of radius $\sqrt{c^2 + R^2}$, but this time consider the contour γ which is the major arc that connects $c - iR$ to $c + iR$. From Cauchy's theorem, we have that

$$I_c(x, R) - \frac{1}{2\pi i}\oint_\gamma \frac{x^s}{s} ds = 1$$

and so

$$\begin{aligned}
|\delta(x) - I_c(x, R)| = |1 - I_c(x, R)| &= \left|-\frac{1}{2\pi i}\oint_\gamma \frac{x^s}{s} ds\right| \\
&\leq \frac{1}{2\pi}\oint_\gamma \frac{x^c}{\sqrt{c^2 + R^2}} ds \\
&< x^c,
\end{aligned}$$

as before. This completes the proof.

Fact (4.1.5)**.** *Let $f(s) = D(a_n, s)$ be a Dirichlet series which is absolutely convergent for Re$(s) > c - \epsilon$. Then for $x \in \mathbb{R} - \mathbb{Z}$,*

$$\sum_{n<x} a_n = \frac{1}{2\pi i} \oint_{(c)} f(s) \frac{x^s}{s} ds.$$

Proof. Since $f(s)$ is uniformly convergent in $\mathrm{Re}(s) > c - \epsilon$, then

$$\begin{aligned}
\frac{1}{2\pi i} \oint_{(c)} f(s) \frac{x^s}{s} ds &= \frac{1}{2\pi i} \oint_{(c)} \sum_{n \geq 1} \left(\frac{x}{n} \right)^s \frac{ds}{s} \\
&= \frac{1}{2\pi i} \sum_{n\geq 1} a_n \oint_{(c)} \frac{(x/n)^s}{s} ds \\
&= \sum_{n \geq 1} a_n \delta \left(\frac{x}{n} \right) \\
&= \sum_{1 \leq n < x} a_n.
\end{aligned}$$

□

Fact (4.1.6)**.** *For $c > 0$ and for any integer $k > 1$, a useful identity is*

$$\frac{1}{2\pi i} \oint_{(c)} \frac{x^s}{s^{k+1}} ds = \begin{cases} \dfrac{1}{k!}(\log x)^k & \text{if } x > 1 \\ 0 & \text{if } 0 \leq x < 1 \end{cases}$$

Proof. For $x > 1$, consider the circle of radius $R > c$ centered at c, and set $\mathcal{C}_R$ to be the left semicircle oriented from $c + iR$ to $c - iR$. Set I_R as the line from $c - iR$ to $c + iR$. By Cauchy's theorem, integrating our function around this contour picks up the residue of the pole at $s = 0$, so we'll calculate that first. Observe that the Laurent series is given by

$$\begin{aligned}
\frac{x^s}{s^{k+1}} &= \frac{1}{s^{k+1}} e^{s \log x} \\
&= \frac{1}{s^{k+1}} \sum_{n \geq 0} \frac{(s \log x)^n}{n!} \\
&= \sum_{n\geq 0} \frac{(\log x)^n}{n!} s^{n-k-1}.
\end{aligned}$$

Thus, the residue of the pole at $s = 0$ is the coefficient on the s^{-1} term in this expansion, which occurs when the index $n = k$. That is, by Cauchy's theorem,

$$\frac{1}{2\pi i} \oint_{I_R + \mathcal{C}_R} \frac{x^s}{s^{k+1}} ds = \mathrm{Res}_{s=0} \left(\frac{x^s}{s^{k+1}} \right) = \frac{(\log x)^k}{k!}.$$

Therefore, it suffices to show that integrating our function along $\mathcal{C}_R$ vanishes as $R \to \infty$. Indeed this is the case:

$$\left|\frac{1}{2\pi i}\oint_{\mathcal{C}_R}\frac{x^s}{s^{k+1}}ds\right| \leq \frac{1}{2\pi}\oint_{\mathcal{C}_R}\frac{x^c ds}{(R-c)^{k+1}} = \frac{Rx^c}{2(R-c)^{k+1}}.$$

For $0 < x \leq 1$, consider the same circle but now set $\mathcal{D}_R$ as the right semicircle oriented from $c - iR$ to $c + iR$. Since the function in question is holomorphic inside the contour $D_R - I_R$, Cauchy's theorem gives that integrating along each of these paths gives the same value. So in a similar way as before, we can show that

$$\left|\frac{1}{2\pi i}\oint_{\mathcal{D}_R}\frac{x^s}{s^{k+1}}ds\right| \leq \frac{1}{2\pi}\oint_{\mathcal{D}_R}\frac{x^c}{R^{k+1}}ds = \frac{x^c}{2R^k},$$

which vanishes as $R \to \infty$. □

Exercise (4.1.8)**.** *For a fixed integer $k > 0$ and for any $c > 0$, show that*

$$\frac{1}{2\pi i}\oint_{(c)}\frac{x^s ds}{s(s+1)\dots(s+k)} = \begin{cases}\frac{1}{k!}\left(1-\frac{1}{x}\right)^k & \text{if } x > 1\\ 0 & \text{if } 0 \leq x \leq 1.\end{cases}$$

Proof. For $x > 1$, consider the circle centered at c of radius $R > c + k$ and set $\mathcal{C}_R$ as the left semicircle oriented from $c - iR$ to $c + iR$. As usual, set I_R as the vertical line from $c - iR$ to $c + iR$. Noticing that the interior of this region contains all of the poles of the function in question, Cauchy's theorem gives

$$\begin{aligned}\frac{1}{2\pi i}\oint_{I_R-\mathcal{C}_R}\frac{x^s ds}{s(s+1)\dots(s+k)} &= \sum_{n=0}^{k}\operatorname{Res}_{s=-n}\left(\frac{x^s}{s(s+1)\dots(s+k)}\right)\\ &= \sum_{n=0}^{k}\frac{x^{-n}}{(-n)(-n+1)\dots(-1)(1)\dots(-n+k)}\\ &= \sum_{n=0}^{k}\frac{x^{-n}}{(-1)^n(n)(n-1)\dots(1)(1)\dots(k-n)}\\ &= \sum_{n=0}^{k}\frac{(-1)^n x^{-n}}{n!(k-n)!}\\ &= \sum_{n=0}^{k}\frac{1}{k!}\binom{k}{n}(-x)^{-n}\\ &= \frac{1}{k!}\left(1-\frac{1}{x}\right)^k.\end{aligned}$$

Therefore, it sufficies to show that integrating this function along $\mathcal{C}_R$ vanishes as $R \to \infty$. Indeed, since $|s(s+1)\dots(s+k)| \geq |s^{k+1}| \geq (R-c)^{k+1}$, we have

$$\left|\frac{1}{2\pi i}\oint_{\mathcal{C}_R}\frac{x^s ds}{s(s+1)\dots(s+k)}\right| \leq \frac{1}{2\pi}\oint_{\mathcal{C}_R}\frac{x^c ds}{(R-c)^{k+1}} = \frac{Rx^c}{2(R-c)^{k+1}},$$

which concludes the first case. Supposing $0 \leq x \leq 1$, consider the same circle and set $\mathcal{D}_R$ to be the right semicircle oriented from $c-iR$ to $c+iR$. Our function is holomorphic inside the region between $\mathcal{D}_R$ and I_R, so by Cauchy's theorem, we need only show that the value of the contour integral along $\mathcal{D}_R$ vanishes as $R \to \infty$. Since $|s(s+1)\dots(s+k)| \geq |s^{k+1}| \geq (R+c)^{k+1}$, we see that

$$\left|\frac{1}{2\pi i}\oint_{\mathcal{D}_R}\frac{x^s ds}{s(s+1)\dots(s+k)}\right| \leq \frac{1}{2\pi}\oint_{\mathcal{D}_R}\frac{x^c ds}{(R+c)^{k+1}} = \frac{Rx^c}{2(R+c)^{k+1}}.$$

□

Exercise (4.1.9)**.** *Let $f(s) = D(a_n, s)$ be a Dirichlet series, absolutely convergent for Re$(s) > c - \epsilon$. Show that for any integer $k \geq 1$,*

$$\frac{1}{x^k}\sum_{n\leq x} a_n(x-n)^k = \frac{k!}{2\pi i}\oint_{(c)}\frac{f(s)x^s}{s(s+1)\dots(s+k)}ds.$$

Proof. Using the uniform convergence of f and the result from the previous exercise, we see

$$\begin{aligned}\frac{k!}{2\pi i}\oint_{(c)}\frac{f(s)x^s}{s(s+1)\dots(s+k)}ds &= \frac{k!}{2\pi i}\oint_{(c)}\sum_{n\geq 0}\frac{a_n n^{-s}x^s}{s(s+1)\dots(s+k)}ds \\ &= \sum_{n\geq 0} a_n \frac{k!}{2\pi i}\oint_{(c)}\frac{(x/n)^s ds}{s(s+1)\dots(s+k)} \\ &= \sum_{n\geq 0} a_n\left(1-\frac{1}{x/n}\right)^k \\ &= \frac{1}{x^k}\sum_{n\leq x} a_n(x-n)^k.\end{aligned}$$

□

Section 4.2: The Prime Number Theorem

For $T > e^2$, set $\sigma_0 = 1 - \frac{1}{\log T}$ and consider the rectangular contour R_T, which has diagonal vertices $\sigma_0 - iT$ and $2 + iT$ and counterclockwise orientation. Of interest is to integrate the function

$$f(s) = -\frac{\zeta'(s)}{\zeta(s)}$$

around this contour. The immediate problem, however, is that this pushes to the left of $\mathrm{Re}(s) = 1$, and we have very limited knowledge about the zero behaviors of $\zeta(s)$, $\zeta'(s)$ in that region. But we do know the following: recall that $f(s) = D(\Lambda(n), s)$, and so by (4.1.5), we can express the partial sum up to some non-integer x as

$$\psi(x) = \sum_{n<x} \Lambda(n) = \frac{1}{2\pi i} \oint_{(2)} f(s) \frac{x^s}{s} ds.$$

In particular, notice that $\psi(x)$ constitutes the integral along the right dimension of R_T as $T \to \infty$. If we could somehow morph the left dimension of R_T so that the only pole of $f(s)$ contained in the modified contour R_T^* is at $s = 1$, then by Cauchy's theorem, we'd have

$$\frac{1}{2\pi i} \oint_{R_T^*} f(s) \frac{x^s}{s} = \mathrm{Res}_{s=1} \left(f(s) \frac{x^s}{s} \right) = x,$$

which is the main term in $\psi(x)$. It would then just be a matter of showing that the error terms (the values upon integrating the top, left, and bottom portions of the contour) vanish as $x \to \infty$. This is the motivation for the next few bounds.

Fact (4.2.1)**.** *Let Re$(s) > 1$. Then*

$$\zeta(s) = \sum_{m=1}^{n-1} \frac{1}{m^s} - \frac{n^{-s}}{2} + \frac{n^{1-s}}{s-1} - s \int_n^\infty \frac{t - [t] - 1/2}{t^{s+1}} dt.$$

Proof. Taking $f(t) = 1/t^s$ and $k = 0$ in Euler-MacLauren summation yields

$$\begin{aligned}\sum_{m=n}^{B} \frac{1}{m^s} &= \int_n^B \frac{dt}{t^s} + \frac{1}{2} \left(\frac{1}{B^s} - \frac{1}{n^s} \right) - s \int_n^B \frac{\{t\} - 1/2}{t^{s+1}} dt \\ &= \frac{1}{1-s} (B^{1-s} - n^{1-s}) + \frac{1}{2} \left(\frac{1}{B^s} - \frac{1}{n^s} \right) - s \int_n^B \frac{t - [t] - 1/2}{t^{s+1}} dt,\end{aligned}$$

whereupon taking $B \to \infty$ gives

$$\sum_{m=n}^{\infty} \frac{1}{m^s} = \frac{n^{1-s}}{s-1} - \frac{n^{-s}}{2} - s \int_n^\infty \frac{t - [t] - 1/2}{t^{s+1}} dt.$$

Thus, we have established the following meromorphic continuation of $\zeta(s)$ to $\mathrm{Re}(s) > 0$:

$$\zeta(s) = \sum_{m=1}^{n-1} \frac{1}{m^s} - \frac{n^{-s}}{2} + \frac{n^{1-s}}{s-1} - s \int_n^\infty \frac{t - [t] - 1/2}{t^{s+1}} dt.$$

□

Fact (4.2.2)**.** *For $s = \sigma + it \in R_T$,*

$$\zeta(s) - \frac{1}{s-1} = O(\log T).$$

Proof. From the previous fact, we can write

$$\zeta(s) - \frac{1}{s-1} = \sum_{m=1}^{n-1} \frac{1}{m^s} + \frac{n^{1-s}-1}{s-1} - \frac{n^{-s}}{2} - s\int_n^\infty \frac{\{x\}-1/2}{x^{s+1}}dx.$$

Applying the triangle inequality gives the bound

$$\left|\zeta(s) - \frac{1}{s-1}\right| \leq \sum_{m=1}^{n-1} \frac{1}{m^\sigma} + \int_1^n \frac{dx}{x^\sigma} + \frac{1}{2n^\sigma} + \frac{|s|}{2}\int_n^\infty \frac{dx}{x^{\sigma+1}}.$$

Since

$$\sum_{m=1}^{n-1} \frac{1}{m^\sigma} + \frac{1}{2n^\sigma} < \int_1^n \frac{dx}{x^\sigma} + 1,$$

we have

$$\begin{aligned} \left|\zeta(s) - \frac{1}{s-1}\right| &\leq 1 + 2\int_1^n \frac{dx}{x^\sigma} + \frac{|s|}{2}\int_n^\infty \frac{dx}{x^{\sigma+1}} \\ &\leq 1 + 2\int_1^n \frac{dx}{x^{\sigma_0}} + \frac{|s|}{2}\int_n^\infty \frac{dx}{x^{\sigma_0+1}}. \end{aligned}$$

□

Estimates on $\zeta(s)$

3 April 2013

Note $\sum_{m=1}^{n} \frac{1}{m^\sigma} + \frac{1}{2n^\sigma} < 1 + \int_1^n \frac{dx}{x^\sigma}$. Thus $|\zeta(s) - \frac{1}{s-1}| \leq 1 + s\int_1^n \frac{dx}{x^{\sigma_0}} + \frac{|s|}{2}\int_n^\infty \frac{dx}{x^{\sigma_0+1}} = 1 + 2[\frac{n^{1-\sigma_0}-1}{1-\sigma_0} + \frac{|s|n^{-\sigma_0}}{2\sigma_0}](*)$.

Note As $\sigma_0 \to 1$, $(*) \to 1 + 2[\log(n) + \frac{|\sigma+it|}{2n}]$ which is minimized in terms of n at the same value of n as $(*)$, namely $\frac{|s|}{4}$.

For $s = \sigma_0 + it$, $\frac{|s|}{4} = \frac{\sigma_0+it}{4} \leq \frac{t}{4} + \frac{1}{4t}$. We will take $n = [t]$. So for $s \in R_T$, $|\zeta(s) - \frac{1}{s-1}| \leq 1 + 2\int_0^T \frac{dx}{x^{\sigma_0}} + \frac{|s|}{2}\int_T^\infty \frac{dx}{x^{\sigma_0+1}} = 1 + 2[\frac{T^{1-\sigma_0}-1}{\frac{1}{\log(T)}}] + \frac{|s|T^{1-\sigma_0}}{2\sigma_0 T} \leq 2T^{1-\sigma_0}\log(T) + \frac{|s|T^{1-\sigma_0}}{2\sigma_0 T} = T^{\frac{1}{\log(T)}}[2\log(T) + \frac{|s|}{2\sigma_0 T}] = e[2\log(T) + \frac{|s|}{2\sigma_0 T}]$. Now for $s \in R_T$, $|s| < 2+T$, and since $\sigma_0 > \frac{1}{2}$ because $T > e^2$, $\frac{|s|}{2\sigma_0} < \frac{2+T}{2\sigma_0} < 2+T$. So $|\zeta(s) - \frac{1}{s-1}| \leq e[2\log(T) + \frac{2+T}{T}] << \log(T)$.

Fact (4.2.3)**.** *For $s \in \partial R_T$, $\zeta(s) = O(\log(T))$.*

Proof. Note For $s = \sigma + it \in \partial R_T$, $|\frac{1}{s-1}| \leq \frac{1}{|\sigma - 1 + it|} \leq \min\{\frac{1}{|\sigma-1|}, \frac{1}{1+1}\}$. From before, we have for $s \in \partial R_T$, $|\zeta(s)| \leq \frac{1}{s-1} + O(\log(T))$. $|\zeta(s) - \frac{1}{s-1}| << O(\log(T)) \implies \zeta(s) = \frac{1}{s-1} + O(\log(T)) \leq \min\{\frac{1}{|\sigma-1|}, \frac{1}{t}\} + O(\log(T))$. On horizontal strips of ∂R_T, $|\zeta(s)| \leq \frac{1}{|T|} + O(\log(T)) << \log(T)$. On vertical strips, $|\zeta(s)| \leq \frac{1}{\sigma - 1} + O(\log(T)) \leq \frac{1}{\frac{1}{\log(T)}} + O(\log(T)) << \log(T)$. □

Exercise (4.2.4)**.** *For $s = \sigma + it$ with fixed $\sigma \geq 1/2$, show that as $|t| \to \infty$,*

$$\zeta(\sigma + it) = O(|t|^{1/2}).$$

Proof. Using the result from (4.2.1) and methods following the proof of (4.2.2), we can express

$$\begin{aligned} |\zeta(\sigma + it)| &\leq 1 + \frac{n^{1-\sigma} - 1}{1 - \sigma} + \frac{n^{1-\sigma}}{(\sigma-1)^2 + t^2} + |\sigma + it| \frac{1}{\sigma n^\sigma} \\ &\leq 1 + \frac{n^{1/2} - 1}{1 - \sigma} + \frac{n^{1/2}}{(\sigma-1)^2 + t^2} + \frac{1}{n^\sigma} + \frac{|t|}{\sigma n^\sigma}. \end{aligned}$$

Here, setting $n = |t| \to \infty$ gives the result. □

Fact (4.2.5)**.** *For $s \in R_T$,*

$$|\zeta'(s) - \frac{1}{(s-1)^2}| = O(\log^2 T)$$

. From 4.2.1,

$$\zeta(s) - \frac{1}{s-1} = \sum_{m=1}^{n-1} \frac{1}{m^s} - \int_1^n \frac{dx}{x^s} - \frac{1}{2n^s} - s \int_n^\infty \frac{\{x\} - \frac{1}{2}}{x^{s+1}} dx$$

. Differentiate with respect to s

$$\zeta'(s) - \frac{1}{(s-1)^2} = \sum_{m=1}^{n-1} \frac{\log m}{m^s} + \int_1^n \frac{\log(x)}{x^s} dx + \frac{\log n}{2n^s} - \int \frac{\{x\} - \frac{1}{2}}{x^{s+1}} + s \int_n^\infty \frac{(\{x\} - \frac{1}{2}) \log x}{x^{s+1}} dx$$

. So, letting $n = [T]$ again,

$$\zeta'(s) + \frac{1}{(s-1)^2} \leq \left(\log T + 2 \log T \int_1^\infty \frac{dx}{x^{\sigma_0}} \right) + \frac{1}{2} \int_T^\infty \frac{dx}{x^{\sigma_0 + 1}} + \frac{|s|}{2} \int_n^\infty \frac{\log x}{x^{\sigma_0 + 1}} dx$$

.

$$u = \log x$$
$$du = \frac{1}{x}$$
$$dv = x^{-1-\sigma_0} dx$$
$$v = \frac{-1}{\sigma_0} x^{-\sigma_0}$$
$$= \log T (1 + 2\int_1^T \frac{dx}{x^{\sigma_0}}) + \frac{1}{2}\left[\frac{1}{\sigma_0 T^{\sigma_0}}\right] + \frac{|s|}{2}\left[\frac{\log T}{\sigma_0 T^{\sigma_0}}\right] + \frac{1}{\sigma_0}\int_1^\infty \frac{dx}{x^{\sigma_0+1}}$$

.

Bounds on zeta and zeta prime

5 April 2013

Bound $\zeta'(s)$
Recall: we have

$$\zeta(s) - \frac{1}{s-1} = \sum_{m=1}^{n-1} \frac{1}{m^s} - \int_1^n \frac{\mathrm{d}x}{x^s} - \frac{1}{2n^s} - s\int_n^\infty \frac{\{x\} - \frac{1}{2}}{x^{s+1}}\mathrm{d}x.$$

Letting $n = [T]$ and applying the triangle inequality a few times, we obtained the bound:

$$\begin{aligned}
|\zeta'(s) + \frac{1}{(s-1)^2}| &\le \log T\left(1 + 2\int_1^T \frac{\mathrm{d}x}{x^{\sigma_0}}\right) + \frac{1}{2}\left(\frac{1}{\sigma_0 T^{\sigma_0}}\right) + \frac{|s|}{2}\left(\frac{\log T}{\sigma_0 T^{\sigma_0}} + \frac{1}{\sigma_0}\int_T^\infty \frac{\mathrm{d}x}{x^{\sigma_0+1}}\right) \\
&\ll \log^2 T + \frac{1}{2}\left(\frac{e}{\sigma_0 T}\right) + \frac{|s|}{2}\left(\frac{e\log T}{\sigma_0 T} + \frac{1}{\sigma_0}\left(-\frac{1}{\sigma_0}x^{-\sigma_0}\Big|_T^\infty\right)\right) \\
&= \log^2 T + \frac{1}{2}\left(\frac{e}{\sigma_0 T}\right) + \frac{|s|}{2}\left(\frac{e\log T}{\sigma_0 T} + \frac{e}{\sigma_0^2 T}\right) \\
&\ll \log^2 T + \frac{\log T}{1 - \log T} + \frac{1}{\left(1 - \frac{1}{\log T}\right)^2} \\
&\ll \log^2 T.
\end{aligned}$$

Using side calculation:

$$\begin{aligned}
\sigma_0 &= 1 - \frac{1}{\log T} \\
T^{\sigma_0} &= e^{\log T\left(1 - \frac{1}{\log T}\right)} \\
&= e^{\log T} e^{-1} \\
&= \frac{T}{e}
\end{aligned}$$

and the facts: $|s| = \sqrt{T^2+4} \ll T$ *and* $T > e^2$.

Fact
For $s = \sigma + it \in \partial R_T, \zeta'(s) = O(\log^2 T)$.

Proof. From before, we have $\zeta'(s) + \dfrac{1}{(s-1)^2} = O(\log^2 T)$
$\Rightarrow \zeta'(s) = \dfrac{1}{(s-1)^2} + O(\log^2 T) \Rightarrow |\zeta'(s)| = \dfrac{1}{(\sigma-1)^2}$.
As before, $\dfrac{1}{(s-1)^2} \leq \min\{\dfrac{1}{(\sigma-1)^2, \frac{1}{t^2}}\}$. So, on the vertical strips of our contour, we have $\sigma \geq 1 - \dfrac{1}{\log T}$ or $= 2$.
This implies

$$|\sigma - 1| \geq \frac{1}{\log T} \Rightarrow \frac{1}{|\sigma-1|} \leq \log T \Rightarrow \frac{1}{|\sigma-1|^2} \leq \log^2 T.$$

Thus, $|\zeta'(s)| \ll \dfrac{1}{|s-1|^2} + \log^2 T \ll \log^2 T$.
On the horizonal strips, $t = T$ so we have $\dfrac{1}{|s-1|^2} < \dfrac{1}{t^2} = \dfrac{1}{T^2} \leq \log^2 T$ thus, we conclude $|\zeta'(s)| \ll \log^2 T$. $\square$

Fact 4.2.7
Let $s = \sigma + it$ *as before. Then, there exist* $c_1, c_2 \in \mathbb{R}_{>0}$ *such that* $1 - \dfrac{c_1}{\log^9 T} \leq \sigma \leq 2$ *and* $1 \leq |t| \leq T$ *then* $|\zeta(s)| > \dfrac{c_2}{\log^7 T}$.

Proof. Recall from exercise 3.2.5, we have $|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \geq 1$ for $\sigma > 1$. This implies $|\zeta(\sigma+it)|^4 \leq \dfrac{1}{|\zeta(\sigma+it)\zeta^3(\sigma)|}$.
Also, recall that $\zeta(\sigma)(\sigma-1)$ is bounded as $\sigma \to 1^+$. Thus, $f(\sigma)(\sigma-1)$ is continuous on $[1,2]$ so this function will acheive a maximum value. Choose this and call it B at σ'.
In particular, this means $B = \zeta(\sigma')(\sigma'-1) \Rightarrow |\zeta(\sigma)(\sigma-1)| \leq B$ for $\sigma \in [1,2] \Rightarrow$
$\dfrac{1}{|\zeta(\sigma)|} \geq \dfrac{|\sigma-1|}{B}$.
By exercise 4.2.3, we have that there exists k' such that
$|\zeta(\sigma+2it)| \leq k' \log T \Rightarrow \dfrac{1}{|\zeta(\sigma+2it)|} \geq \dfrac{k}{\log T}$.
So, we have, $|\zeta(\sigma+2it)|^4 \geq \dfrac{k_1(\sigma-1)^3}{\log T}$ where $\dfrac{1}{k'} = k$ and $k_1 = \dfrac{k}{B^3}$.
Then, $|\zeta(\sigma+2it)|^4 \geq \dfrac{k_1(\sigma-1)^3}{\log T}$.

If $1+\dfrac{c_1}{\log^9 T} \le \sigma \le 2$, then $|\zeta(\sigma+2it)|^4 \gg \dfrac{k_1\left(\dfrac{c_1^3}{\log^{27} T}\right)}{\log T} = \dfrac{k_1 c_1^3}{\log^{28} T}$.
Thus, $|\zeta(s)| \gg \dfrac{1}{\log^7 T}$.
Now, we will use MVT to extend into the region of interest. Choose $s' = \sigma' + it$ where $\sigma' = 1 + \dfrac{c_1}{\log^9 T}$ and $1 \le t \le T$.
Then, by MVT,

$$|\zeta(\sigma'+it) - \zeta(\sigma+it)| = |\sigma - \sigma'||\zeta'(c)|$$

for some $\sigma' < c < \sigma$ Then, by 4.2.5, we have

$$|\zeta(\sigma'+it) - \zeta(\sigma+it)| \ll |\sigma' - \sigma|(\log^2 T.)$$

So, we can say

$$|\zeta(s)| = |\zeta(\sigma'+it)| + O((\sigma-\sigma')\log^2 T). \tag{$\star$}$$

Choosing $1-\dfrac{c_1}{\log^9 T} \le \sigma \le 1+\dfrac{c_1}{\log^9 T}$ so that $|\sigma-\sigma'| \le \dfrac{2c_1}{\log^7 T} \Rightarrow |\sigma-\sigma'|\log^2 T \le \dfrac{2c_1}{\log^7 T}$.
Thus, for c_1 sufficiently small (smaller than the constant implied in ($\star$)), we obtain $|\zeta(s)| \gg \dfrac{1}{\log^7 T}$. □

With this, we can prove the prime number theorem.

Theorem (4.2.9)**.** *There is a positive constant c so that*

$$\psi(x) = x + O(x\exp(-c(\log x)^{1/10})).$$

Remark. Note that this is indeed enough as the error term is $o(x)$.

Proof. Let x be exactly 1/2 more than any natural number. Using the representation $\psi(x) = -\zeta'(s)/\zeta(s)$, we invoke (4.1.5) and write

$$\psi(x) = \frac{1}{2\pi i}\oint_{(a)} \frac{-\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds,$$

for any choice of $a > 1$. We'll take $a = 1 + c/\log^9 T$, with $T \ge 1$ to be chosen later. Seeing as $\delta(x/n) > 1$, multiplying the result of (4.1.4) by $\psi(x)$ gives the approximation

$$\psi(x) = \frac{1}{2\pi i}\int_{a-iT}^{a+iT} \frac{-\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds + O\left(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^a \Lambda(n)\min\left(1, \frac{1}{T|\log x/n|}\right)\right).$$

The bulk of the work comes in estimating this error term, which we'll do in cases. First, if $n < \frac{x}{2}$ or if $n > \frac{3x}{2}$, then the smallest $|\log \frac{x}{n}|$ can be is $\log \frac{3}{2}$, which doesn't depend on x. In particular, then,

$$\begin{aligned}\frac{\Lambda(n)}{n^a} x^a \min\left(1, \frac{1}{T|\log x/n|}\right) &\leq \frac{1}{n^{a-1}} x^a \min\left(1, \frac{1}{T \log 3/2}\right) \\ &\ll \frac{x^a}{(a-1)T} \\ &\ll \frac{x^a \log^9 T}{T}.\end{aligned}$$

Now, for those n such that $\frac{x}{2} < n < \frac{3x}{2}$, set $z = 1 - \frac{n}{x}$ so that

$$\log \frac{x}{n} = -\log(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \dots.$$

Again, we need to determine how small the log term in the denominator can become. Since $|z| \leq 1/2$, this implies that

$$\left|\log \frac{x}{n}\right| = |z| \left|\left(1 + \frac{z}{2} + \frac{z^2}{3} + \dots\right)\right| \geq \frac{3}{4}|z|.$$

Since this bound depends on x, we'll have to treat the entire sum at one time. In particular, we have

$$\begin{aligned}\sum_{\frac{x}{2} < n < \frac{3x}{2}} \frac{\Lambda(n)}{n^a} x^a \min\left(1, \frac{1}{T|\log x/n|}\right) &\leq \sum_{\frac{x}{2} < n < \frac{3x}{2}} \frac{\Lambda(n)}{n^a} \cdot \frac{x^a}{T \cdot \frac{3}{4}\left|1 - \frac{n}{x}\right|} \\ &\leq \sum_{\frac{x}{2} < n < \frac{3x}{2}} \frac{\log n}{\left(\frac{x}{2}\right)^a} \cdot \frac{x^a}{T \cdot \frac{3}{4}\left|1 - \frac{n}{x}\right|} \\ &\leq \sum_{\frac{x}{2} < n < \frac{3x}{2}} \log\left(\frac{3x}{2}\right) \frac{2^a}{T \cdot \frac{3}{4}\left|1 - \frac{n}{x}\right|} \\ &\leq \sum_{\frac{x}{2} < n < \frac{3x}{2}} \log\left(\frac{3x}{2}\right) \frac{x 2^a}{T \cdot \frac{3}{4}|x-n|} \\ &\ll \frac{x}{T} \log x \sum_{\frac{x}{2} < n < \frac{3x}{2}} \frac{1}{|x-n|} \\ &\ll \frac{x}{T} (\log x)^2.\end{aligned}$$

Therefore, we have rewritten our central approximation as

$$\psi(x) = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{-\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds + O\left(\frac{x^a \log^9 T}{T} + \frac{x \log^2 x}{T}\right).$$

Recall our rectangular contour, R_T, with diagonal vertices $b - iT$ and $a + iT$ and counterclockwise orientation. Upon setting $b = 1 - \frac{c}{\log^9 T}$, Cauchy's residue theorem gives

$$\frac{1}{2\pi i}\oint_{a-iT}^{a+iT}-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds = x-\frac{1}{2\pi i}\left(\oint_{a+iT}^{b+iT}-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds+\oint_{b+iT}^{b-iT}-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds+\oint_{b-iT}^{a-iT}-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds\right).$$

These integrals are easily bounded with the help of (4.2.5) through (4.2.7). Observe that for $s \in R_T$, we have

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| = \frac{O(\log^2 T)}{O(1/\log^7 T)} = O\left(\log^9 T\right).$$

The work in bounding the $\frac{x^s}{s}$ bit has already been done in the proof of case 3 of (4.2.1), and so in particular we have

$$\begin{aligned}
\left|\frac{1}{2\pi i}\oint_{a+iT}^{b+iT}-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds\right| &\ll \frac{x^a\log^9 T}{T}\\
\left|\frac{1}{2\pi i}\oint_{b-iT}^{a-iT}-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds\right| &\ll \frac{x^b\log^9 T}{T} \ll \frac{x^a\log^9 T}{T}\\
\left|\frac{1}{2\pi i}\oint_{b+iT}^{b-iT}-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds\right| &\ll x^b\log^9 T\cdot\frac{T\log^9 T}{\log^9 T - c} \ll x^b\log^{10} T,
\end{aligned}$$

where the final bound is because

$$\frac{T\log^9 T}{\log^9 T - c} \ll \frac{T\log^9 T}{T\log^8 T} \ll \log T.$$

Therefore, we have established

$$\psi(x) = x + O\left(\frac{x^a\log^9 T}{T} + x^b\log^{10} T + \frac{x\log^2 x}{T}\right).$$

Setting T so that

$$2c\log x = \log^{10} T$$

gives the result. □

Ch. 5: Functional Equations

5.1: Poisson Summation

Theorem (Fejér). *Let $f(x)$ be a function of a real variable that is bounded, measurable, and periodic with period 1. Set the Fourier coefficients of f to be*

$$c_n = \int_0^1 f(x)e^{-2\pi i n x}dx$$

and also define

$$S_N(x) = \sum_{|n| \le N} c_n e^{2\pi i n x}.$$

If f is continuous at x_0 and if the sequence $(S_N(x_0))$ converges, then

$$f(x_0) = c_0 + \sum_{n=1}^{\infty} (c_n e^{2\pi i n x_0} + c_{-n} e^{-2\pi i n x_0}).$$

Furthermore, if f is continous and $\sum_{n=-\infty}^{\infty} |c_n|$ converges, then

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}.$$

Definition. If $F(x)$ is continous with $\int_{-\infty}^{\infty} |F(x)| dx < \infty$, we define the *Fourier transform of F* to be

$$\widehat{F}(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x u} dx.$$

One can show that $\widehat{F}(u)$ is also continuous, and that $\hat{\hat{F}}(u) = F(-x)$.

Exercise (5.1.1)**.** *For $Re(c) > 0$, let $F(x) = e^{-c|x|}$. Show that*

$$\hat{F}(u) = \frac{2c}{c^2 + 4\pi^2 u^2}.$$

Proof. We calculate

$$\begin{aligned}
\hat{F}(u) &= \int_{-\infty}^{\infty} e^{-c|x| - 2\pi i x u} dx \\
&= \int_{-\infty}^{0} e^{x(c - 2\pi i u)} dx + \int_{0}^{\infty} e^{-x(c + 2\pi i u)} dx \\
&= \frac{e^x}{c - 2\pi i u}\Big|_{-\infty}^{0} - \frac{e^{-x}}{c + 2\pi i u}\Big|_{0}^{\infty} \\
&= \frac{2c}{c^2 + 4\pi^2 u^2}.
\end{aligned}$$

□

Exercise (5.1.2)**.** *For $F(x) = e^{-\pi x^2}$, show that $F\hat{(}u) = e^{\pi u^2}$.*

Theorem (5.1.3: Poisson Summation)**.** *Let $F \in L^1(\mathbb{R})$. Suppose that*

$$\sum_{n \in \mathbb{Z}} F(n + v)$$

converges absolutely and uniformly in v, and that

$$\sum_{m\in\mathbb{Z}} |\hat{F}(m)|$$

converges. Then

$$\sum_{n\in\mathbb{Z}} F(n+v) = \sum_{m\in\mathbb{Z}} \hat{F}(m)e^{2\pi inm}.$$

Corollary. *Taking $v=0$ in the previous result gives*

$$\sum_{n\in\mathbb{Z}} F(n) = \sum_{m\in\mathbb{Z}} \hat{F}(m).$$

Exercise (5.1.5)**.** *With F as before, show that*

$$\sum_{n\in\mathbb{Z}} F\left(\frac{n+v}{t}\right) = \sum_{m\in\mathbb{Z}} |t|\hat{F}(mt)e^{2\pi imtv}.$$

Exercise (5.1.6)**.** *Show that*

$$\frac{e^c+1}{e^c-1} = \sum_{n=-\infty}^{\infty} \frac{2c}{c^2+4\pi^2n^2}.$$

Proof. Using (5.1.1) and the corollary above, it suffices to show

$$\begin{aligned}\sum_{n\in\mathbb{Z}} e^{-c|n|} &= 2\sum_{n=0}^{\infty} e^{-cn} - 1\\ &= 2\left(\frac{1}{1-e^{-c}}\right) - 1\\ &= \frac{e^c+1}{e^c-1}.\end{aligned}$$

□

Exercise (5.1.7)**.** *Show that*

$$\sum_{n\in\mathbb{Z}} e^{-(n+\alpha)^2\pi/x} = x^{1/2}\sum_{n\in\mathbb{Z}} e^{-n^2\pi x+2\pi in\alpha}.$$

Corollary. *Taking $\alpha=0$ in the previous exercise gives*

$$\sum_{n\in\mathbb{Z}} e^{-n^2\pi/x} = x^{1/2}\sum_{n\in\mathbb{Z}} e^{-n^2\pi x}.$$

5.2 Riemann Zeta Function

Definition. Define $\theta(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}$. Taking $w = iy$ and $w(y) = \theta(iy)$, then by Thereom 5.18, $w\left(\frac{1}{x}\right) = x^{\frac{1}{2}} w(x)$.

<u>*Recall:*</u>

$$\Gamma(s) = \int_0^{infty} e^{-t} t^{s-1} \mathrm{d}t \mathrm{Re}(s) > 0.$$

So,

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} e^{-t} t^{\frac{s}{2}-1} \mathrm{d}t$$

Take $t = n^2\pi x, \mathrm{d}t = n^2\pi \mathrm{d}x$. *Then,*

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} e^{-n^2\pi x} n^{s-2} x^{\frac{s}{2}-1} n^2 \pi \mathrm{d}x$$

This implies:

$$\pi^{-\frac{-s}{2}} \Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^{\infty} e^{-n^2\pi x} x^{\frac{s}{2}-1} \mathrm{d}x.$$

For Re$(s) > 1$, *we can sum over all* $n \geq 1$ *so that*

$$\pi^{\frac{-s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1} \left(\sum_{n=1}^{\infty} e^{-n^2\pi x}\right) \mathrm{d}x.$$

Note: The sum is absolutely convergent which justifies the swapping of the integral and the summation. Noting that $\sum_{n=1}^{\infty} e^{-n^2\pi x} = \frac{w(x)-1}{2} \overset{set}{=} W(x)$ *so* $W\left(\frac{1}{x}\right) = x^{\frac{1}{2}} W(x) + \frac{x^{\frac{1}{2}}-1}{2}$,

$$\begin{aligned}
\pi^{\frac{-s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) &= \int_0^{\infty} x^{\frac{s}{2}-1} W(x) \mathrm{d}x \\
&= \int_1^{\infty} x^{\frac{s}{2}-1} W(x) \mathrm{d}x + \int_1^{\infty} x^{\frac{s}{2}-1} W\left(\frac{1}{x}\right) \mathrm{d}x \\
&= \int_1^{\infty} x^{\frac{s}{2}-1} W(x) \mathrm{d}x + \int_1^{\infty} \left(x^{\frac{s}{2}} W(x) + \frac{x^{\frac{1}{2}}-1}{2}\right) x^{\frac{s}{2}-1} \mathrm{d}x \\
&= \int_1^{\infty} x^{\frac{s}{2}-1} W(x) \mathrm{d}x + \int_1^{\infty} W(x) x^{\frac{1-s}{2}} \frac{\mathrm{d}x}{x} + \frac{1}{2}\int_1^{\infty} x^{\frac{-(s+1)}{2}} \mathrm{d}x - \frac{1}{2}\int_1^{\infty} x^{-\frac{s}{2}-1} \mathrm{d}x \\
&= \int_1^{\infty} W(x) \left(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right) \frac{\mathrm{d}x}{s} + \frac{1}{s-1} - \frac{1}{s}.
\end{aligned}$$

This is meromorphic on the entire complex plane with simple poles at $s = 0$ *and* $s = 1$. *Also, note that this gives the same value for* s *and* $s-1$ *(around*

$Re(s) = \frac{1}{2}$). *Moreover,*
$\zeta(s) = \frac{1}{2}s(s-1)\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ *is analytic on* $\mathbb{C}$ *and* $\zeta(1-s) = \zeta(s)$.