1. A lake, with volume $V = 100$ km^3, is fed by a river at a rate of r km^3/yr. In addition, there is a factory on the lake that introduces a pollutant into the lake at the rate of p km^3/yr. There is another river that is fed by the lake at a rate that keeps the volume of the lake constant. This means that the rate of flow from the lake into the outlet river is $(p+r)$ km^3/yr. Let $x(t)$ denote the volume of the pollutant in the lake at time t. Then $c(t) = x(t)/V$ is the concentration of the pollutant.

   (a) Show that, under the assumption of immediate and perfect mixing of the pollutant into the lake water, the concenetration satisfies the differential equation

   $$c' + \frac{p+r}{V}\,c = \frac{p}{V}\,.$$

   (b) It has been determined that a concentration of over 2% is hazardous for the fish in the lake. Suppose that $r = 50$ km^3/yr, $p = 2$ km^3/yr, and the initial concentration of pollutant in the lake is zero. How long will it take the lake to become hazardous to the health of the fish?

   (c) Suppose that the factory from parts (a) and (b) stops operating at time $t = 0$, and that the concentration of pollutant in the lake was 3.5% at that time. Approximately how long will it take before the concentration falls below 2% and the lake is no longer hazardous for the fish?

2. Consider two tanks, labeled tank A and tank B for reference. Tank A contains 100 gal of solution in which is dissolved 20 lb of salt. Tank B contains 200 gal of solution in which is dissolved 40 lb of salt. Pure water flows into tank A at a rate of 5 gal/sec. There is a drain at the bottom of tank A, and the solution leaves tank A via this drain at a rate of 5 gal/sec and flows immediately into tank B at the same rate. A drain at the bottom of tank B allows the solution to leave tank B at a rate of 2.5 gal/sec. What is the salt content in tank B at the precise moment that tank B contains 250 gal of solution.

3. For each of the second-order differential equations below, decide whether the equation is linear or nonlinear. If the equation is linear, state whether the equation is homogeneous or inhomogeneous.

   (a) $y'' + 3y' + 5y = 3\cos 2t$

   (b) $t^2y'' = 4y' - \sin t$

   (c) $t^2y'' + (1-y)y' = \cos t$

   (d) $ty'' + (\sin t)y' = 4y - \cos 5t$

   (e) $t^2y'' + 4yy' = 0$

   (f) $y'' + 4y' + 7y = 3e^{-t}\sin t$

   (g) $y'' + 3y' + 4\sin y = 0$

   (h) $(1-t^2)y'' = 3y$

4. Find the general solution to the following 2$^{\text{nd}}$ order linear homogeneous ODEs.

(a) $y'' + 5y' + 6y = 0$

(b) $y'' + y' - 12y = 0$

(c) $y'' + 4y' + 5y = 0$

(d) $y'' + 2y = 0$

(e) $y'' - 4y' + 4y = 0$

(f) $4y'' + 12y' + 9y = 0$

5. In this problem, we will find all solutions to the initial value problem $y'' = \lambda y$, $y(0) = y(\pi) = 0$, where λ is a constant. This equation will turn up later when we study PDEs.

   (a) First, suppose that $\lambda = 0$. That is, solve $y'' = 0$, $y(0) = y(\pi) = 0$.

   (b) Next, suppose $\lambda = \omega^2 \geq 0$.

      (i) Solve the initial value problem $y'' = \omega^2 y$, $y(0) = y(\pi) = 0$.

      (ii) Let $u_1(t) = \cosh \omega t = \frac{e^{\omega t} + e^{-\omega t}}{2}$ and $u_2(t) = \sinh \omega t = \frac{e^{\omega t} - e^{-\omega t}}{2}$. Show that $u_1(t)$ and $u_2(t)$ both solve $y'' = \omega^2 y$, and use this to write the general solution of this differential equation.

      (iii) Solve the initial value problem from part (i) again, but this time, start by using the general solution you found in part (ii) (instead of exponentials).

   (c) Finally, suppose $\lambda = -\omega^2 < 0$. That is, solve $y'' = -\omega^2 y$, $y(0) = y(\pi) = 0$.

   (d) Using your results from parts (a)–(c), describe all solutions to the initial value problem $y'' = \lambda y$, $y(0) = y(\pi) = 0$. What are the possibile values for λ?