MTH/SC 208: Differential Equations

1: Intro to ODEs:

Q: What is a Differential Equation?

A: An equation involving a function & its derivatives.

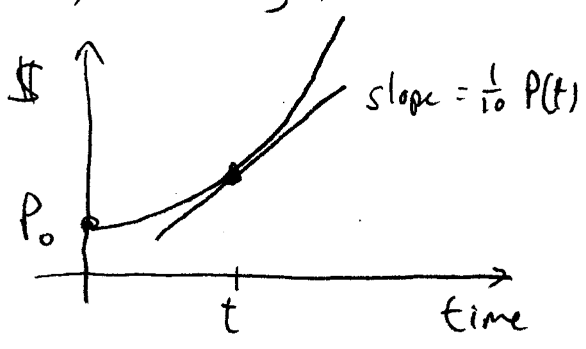
Examples:

- $\boxed{\text{Finance}}$ The $\overbrace{\text{rate of growth of an investment}}^{P'(t)}$ is proportional to the $\underbrace{\text{amount of the investment}}_{P(t)}$.

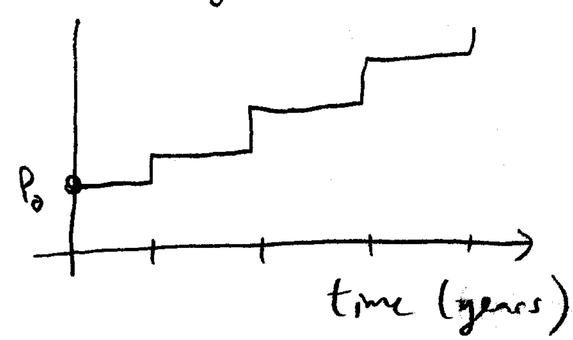
$P'(t) = r\,P(t)$ (often, just write $P' = rP$).

e.g, A mutual fund grows at a 10% rate.

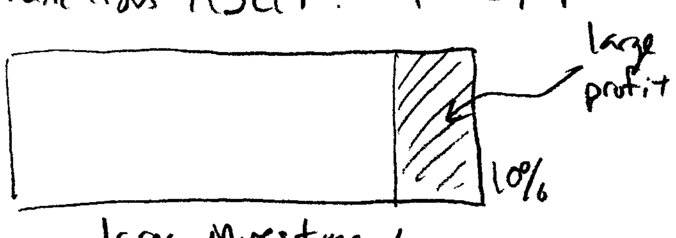
Note: We assume that interest is compounded continuously, i.e, at any point in time, the rate of change is $\frac{1}{10}P(t)$.

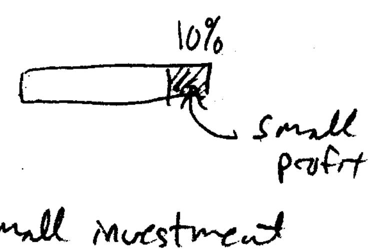


vs.



Big idea: Rate of change of a function is proportional to the function itself: $f' = rf$





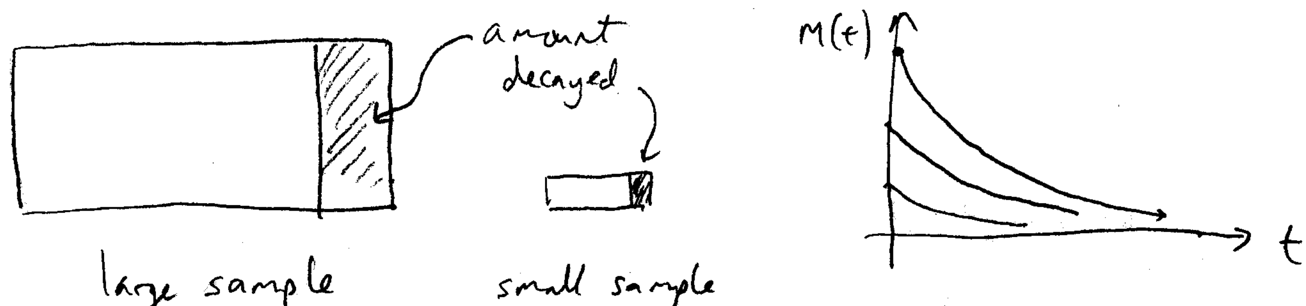
• Biology: A colony of rabbits grows at a rate proportional to its size.

$P'(t) = k\,P(t)$. **Note:** $k > 0$ (why?)

Note: It can't keep doubling forever. This is just a model, good for small t.

e.g., 2 rabbits, 4 rabbits, 8 rabbits, 16 rabbits, etc.

• Chemistry: A radioactive substance decays at a rate proportional to how much is remaining.



$m'(t) = k\,m(t)$. **Note:** $k < 0$ (why?)

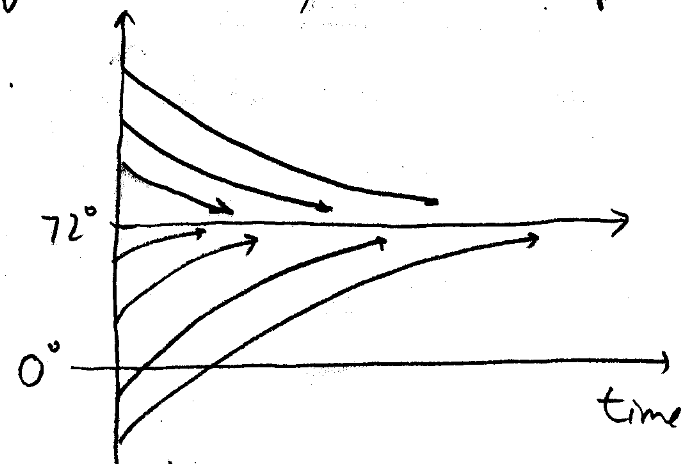
Sample question: If there are 30 grams initially, and 20 grams after one year, what is the half-life?

Think: • Is it even clear that "half-life" is well-defined?

• Compare this to investments.

• Physics: The temperature of a cup of coffee cools at a rate proportional to: "(temp. of coffee) − (ambient temp.)"

Think: Imagine putting a cup of 75° water, and a cup of 200° water, in a 72° room.



$T'(t) = k\,(72 - T(t)), \quad k > 0$

This is decay towards a limiting value.

$T(t) = 72$ is a steady-state (constant) solution.

What else exhibits this behavior in nature (approximately)?

- Earth's population
- Velocity of a falling object with air resistance.

(here, "terminal velocity" plays the role of ambient temp.)

But notice! Population growth is a little different.

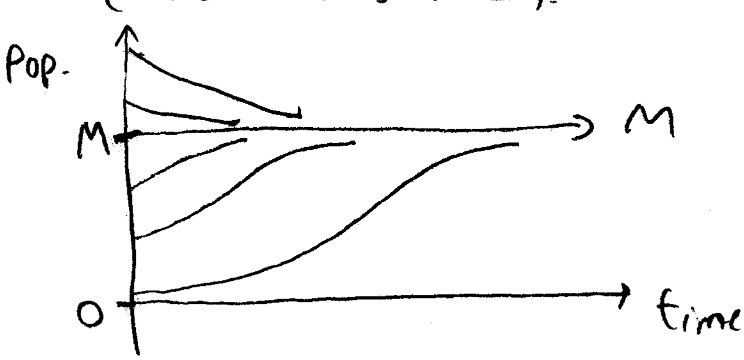
When population is small, it grows exponentially (recall bunnies)

When population is large, it "decays" ⟶ carrying capacity.

Q: How do we put these two together?

A: Logistic equation: $P'(t) = \underbrace{r\left(1 - \frac{P(t)}{M}\right)}_{\text{decay} \to M} \underbrace{P(t)}_{\text{exp. growth}}$

(more on this later).

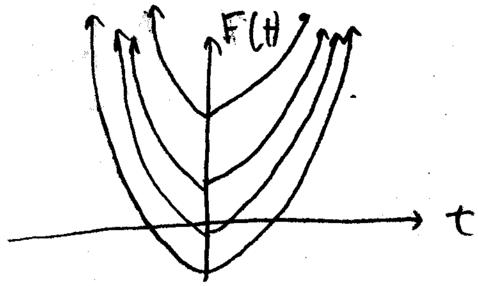


Recall integral calculus.

Q: What is the antiderivative of $f(t) = 2t$?

A: $F(t) = t^2 + C$

Graphically:



All of these have derivative $f(t) = 2t$.

Q: The velocity of a car is $x'(t) = 2t$.

How far from home is it after t hours?

A: $X(t) = t^2 + C$

initial distance from home ("initial condition")

An investment takes 5 years to double.

Q: How much do we have after 8 years?

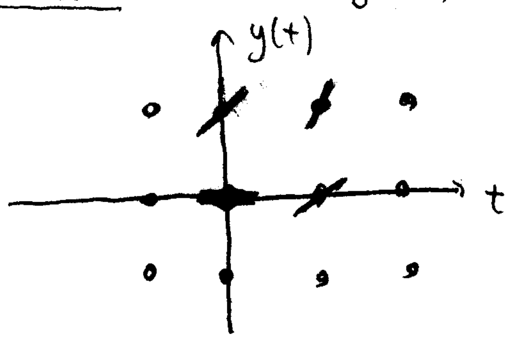
A: We don't know until we specify <u>how much we had initially</u>.

<u>Plotting solutions to "ordinary differential equations" (ODE's)</u>

Consider the ODE: $y' = 2y + t$.

We don't know how to solve it (yet), but we can still visualize the solutions.

<u>Method 1</u>: On a grid, draw the <u>slope field</u>, point-by-point.



$y' = 2y + t$

slope

$(0,0): \; y' = 0$

$(0,1): \; y' = 2$

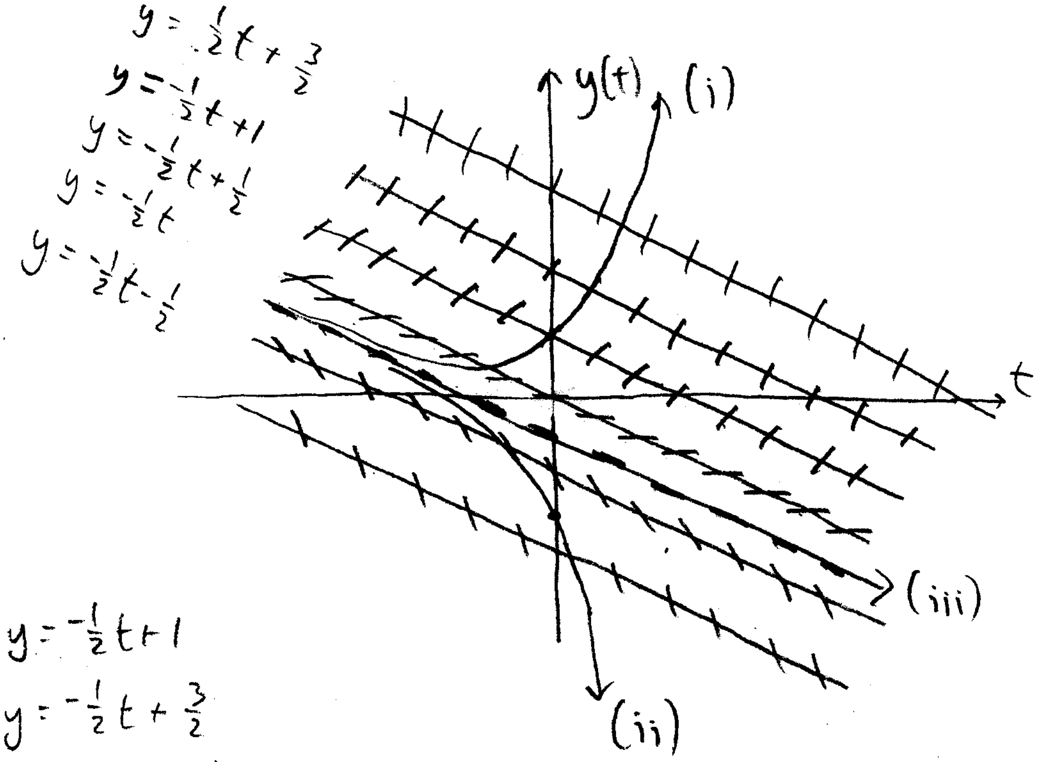
$(1,0): \; y' = 1$

$(1,1): \; y' = 3$

<u>Method 2:</u> Isoclines. (Used to sketch the <u>slope field</u> of an ODE).

<u>Def:</u> An <u>isocline</u> is a line or curve on which $y' = \text{const}$.

<u>Example:</u> $\boxed{y' = 2y + t}$



<u>Q</u>: when is $y' = 0$?

<u>A</u>: when $2y + t = 0$

i.e, $y = -\frac{1}{2}t$.

<u>Q</u>: When is $y' = 1$?

<u>A</u>: When $2y + t = 1$

i.e, $y = -\frac{1}{2}t + \frac{1}{2}$

Continuing... $y' = 2 \Rightarrow y = -\frac{1}{2}t + 1$

$y' = 3 \Rightarrow y = -\frac{1}{2}t + \frac{3}{2}$

$y' = -\frac{1}{2} \Rightarrow y = -\frac{1}{2}t - \frac{1}{4}$

<u>Exercise</u>: Sketch the solution curves satisfying:

(i) $y(0) = 1$, (ii) $y(0) = -\frac{3}{4}$ (iii) $y(1) = -\frac{3}{4}$

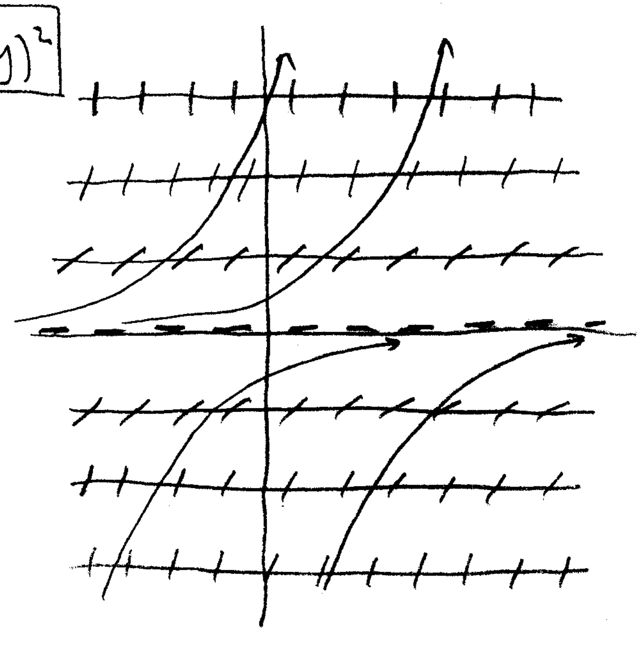
<u>Example</u>. Sketch the solutions of $\boxed{y' = (y)^2}$

$y' = 0 \Rightarrow y = 0$

$y' = 1 \Rightarrow y = \pm 1$

$y' = 4 \Rightarrow y = \pm 2$

$y' = -1 \Rightarrow \emptyset$.



<u>Method 3</u>: A shortcut when the ODE is <u>autonomous</u>;

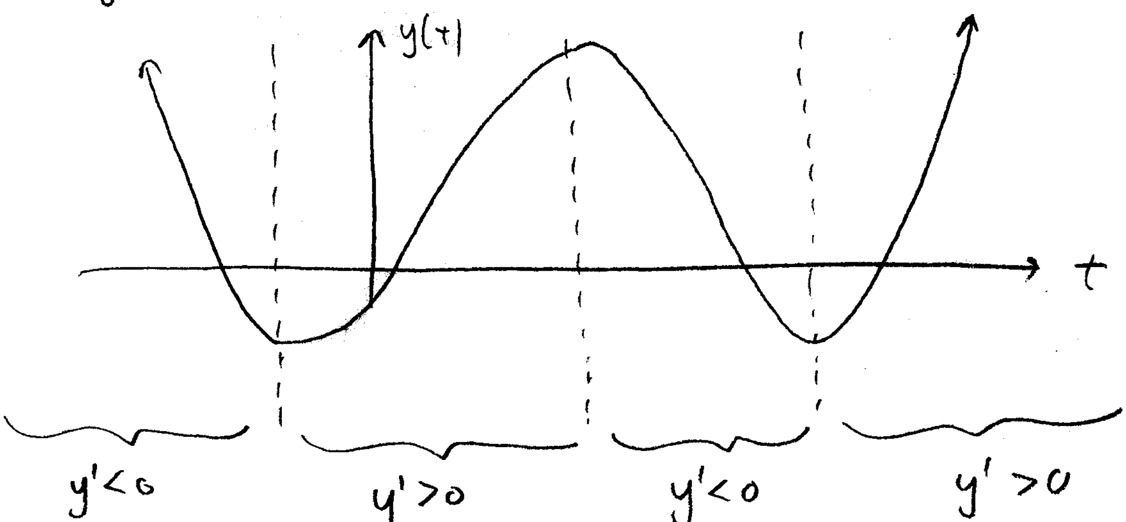
i.e., $\boxed{y' = f(y)}$ for some function f.

e.g.: autonomous: $y' = 3y$ non autonomous: $y' = 3t$

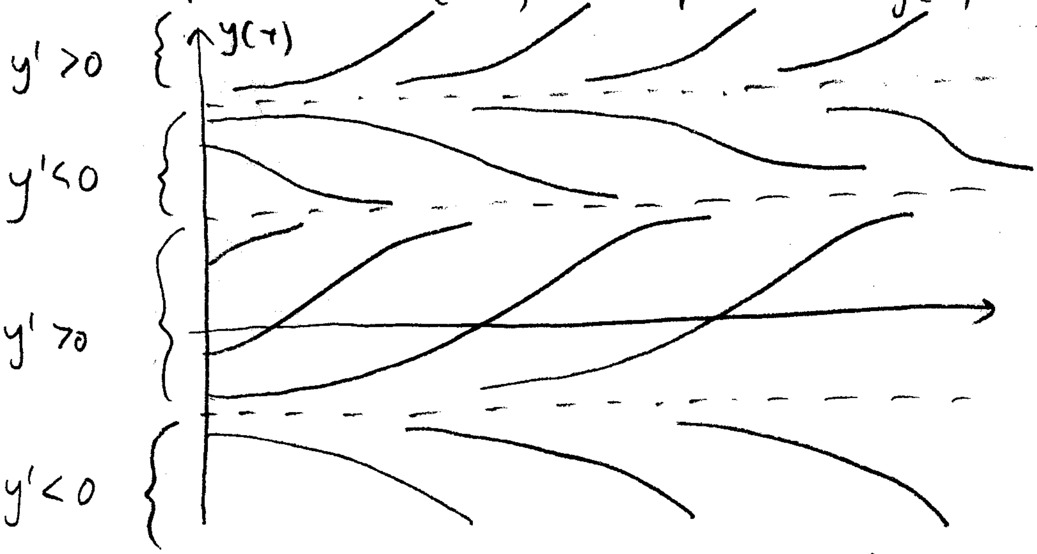
$y' = 2y - 3$ $y' = 2y - 3t$

$y' = y \cdot \sin y - 1$ $y' = t \sin y - 1$

<u>Motivation</u>: Recall basic calculus: Suppose $y'(t) = (t+1)(t-2)(t-4)$.
Plot $y(t)$ (up to a vertical shift).



<u>Now, in ODEs</u>: Suppose $y' = (y+1)(y-2)(y-4)$.
Plot the slope field (i.e, all possible $y(t)$'s).



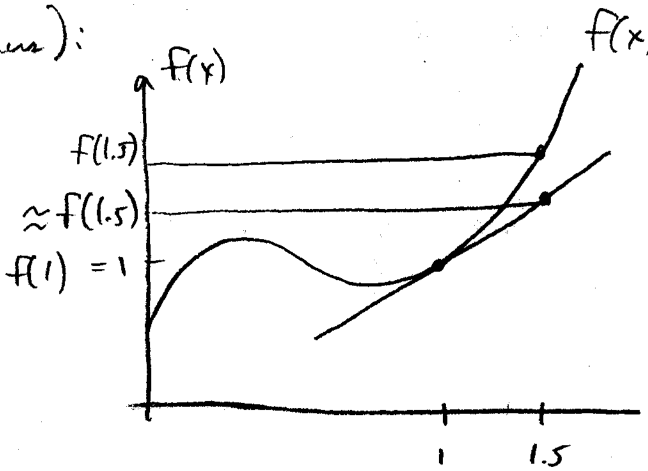
* y' doesn't depend on t; it is <u>autonomous</u>. This was easier than isoclines!

Approximating solutions to ODE's:

Motivation (from single variable calculus):

Suppose $f(1)=1$ and $f'(1)=\frac{1}{2}$.

Use the tangent line to $f(x)$ at $x=1$ to approximate $f(1.5)$.



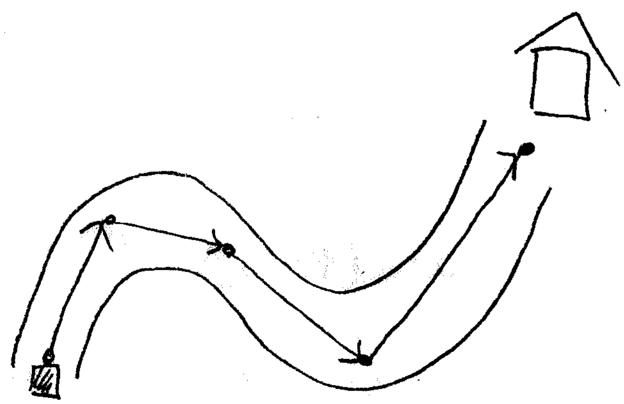
Now in ODE's: Consider the ODE $y'=y-t$ and say $y(1)=1$.

Can we approximate $y(1.5)$?

We'll use a technique called **Euler's method.**

Pictorially: Suppose we want to steer a robot down a winding path.

This is what we'll do on the slopefield of $y'=y-t$.



We'll use a *stepsize* of $h=0.1$.

Again, $\boxed{y'=y-t,\ y(1)=1}$

Start at $(t_0, y_0)=(1,1)$.

Compute slope: $y_0'=y_0-t_0=1-1=0$.

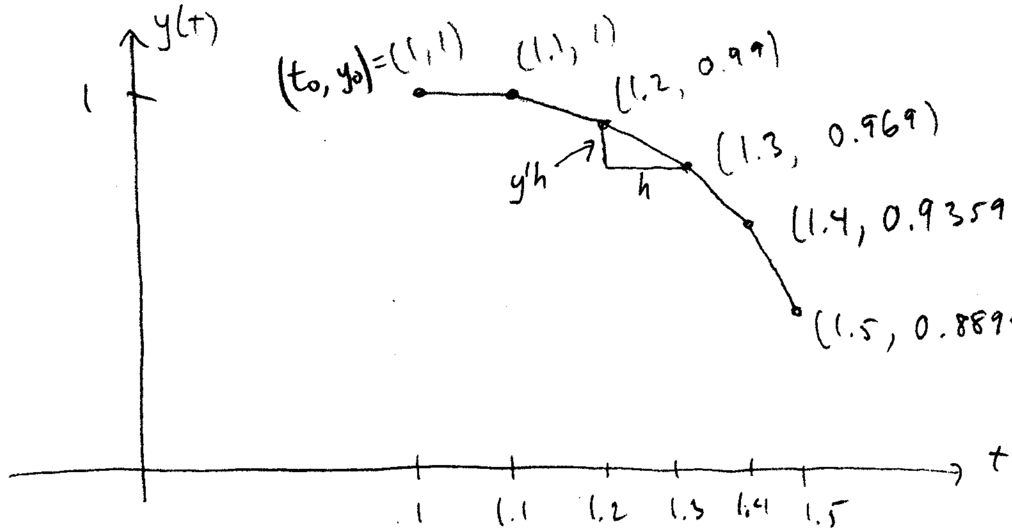
Next point: $(t_1, y_1) = (1.1,\ 1 + y_0' h) = (1.1,\ 1 + 0(0.1)) = (1.1,\ 1)$

Compute slope: $y_1' = y_1 - t_1 = 1 - 1.1 = -0.1$

Next point: $(t_2, y_2) = (1.2,\ 1 + (-0.1)(0.1)) = (1.2,\ 0.99)$

Compute slope: $y_2' = y_2 - t_2 = 0.99 - 1.2 = -0.21$

⋮



⋮

Next point: $(t_3, y_3) = (1.3,\ 0.99 + (-0.21)(0.1)) = (1.3,\ 0.969)$

Recompute slope: $y_3' = y_3 - t_3 = 0.969 - 1.3 = -0.331$.

Next point: $(t_4, y_4) = (1.4,\ 0.969 + (-0.331)(0.1)) = (1.4,\ 0.9359)$

Recompute slope: $y_4' = y_4 - t_4 = 0.9359 - 1.4 = -0.4641$.

Next point $(t_5, y_5) = (1.5,\ 0.9359 + (-0.4641)(0.1)) = (1.5,\ 0.88949)$.

* We've calculated that $\boxed{y(1.5) \approx 0.88949}$.

Note: The actual value is $y(1.5) = -e^{0.5} + 2.5 \approx 0.85128$.

Summary of Euler's method:

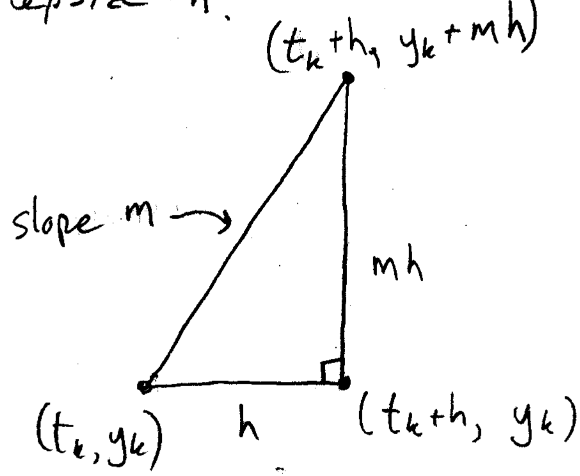
Given $y' = f(t, y)$, $y(t_0) = y_0$, stepsize h.

$(t_1, y_1) = (t_0 + h,\ y_0 + f(t_0, y_0) \cdot h)$

$(t_2, y_2) = (t_1 + h,\ y_1 + f(t_1, y_1) \cdot h)$

$\vdots$

$(t_{k+1}, y_{k+1}) = (t_k + h,\ y_k + f(t_k, y_k) \cdot h)$



Remark: There are "better" approximation methods (e.g., Runge-Kutta), but this one is the most straightforward.