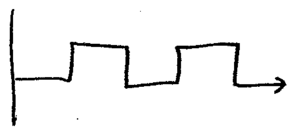
5. Laplace transforms

Laplace transforms are:

- Used to solve linear ODE's
- Useful when the forcing term is discontinuous,

  e.g., step function  (Think Force being turned on/off.)

Big idea:

$$
\begin{tikzcd}
\text{ODE} \arrow[rrr, "\mathcal{L}"] \arrow[d, dashed] & & & \text{Algebraic equation} \arrow[d] \\
\text{Sol'n to ODE} & & & \text{sol'n to algebraic eq'n} \arrow[lll, "\mathcal{L}^{-1}"']
\end{tikzcd}
$$

The Laplace transform is an <u>operator</u>: it inputs a function, and outputs a function.

<u>Def</u>: Suppose $f(t)$ is defined for $0 < t < \infty$. The <u>Laplace transform</u> of f is the function $\mathcal{L}(f)$ where

$$\boxed{\mathcal{L}\big(f(t)\big)(s) := F(s) = \int_0^\infty f(t)\, e^{-st}\, dt, \quad s > 0}$$

Often, we denote $\mathcal{L}(f)$ by F, i.e., $f \xmapsto{\mathcal{L}} F$

<u>Recall</u>: $\displaystyle\int_0^\infty f(t)\, dt = \lim_{T\to\infty} \int_0^T f(t)\, dt.$

Example: Compute $\mathcal{L}(F)$, where $f(t) = e^{at}$.

$$F(s) = \int_0^\infty e^{at} e^{-st}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty$$

$$= \lim_{T\to\infty} \frac{e^{-(s-a)t}}{-(s-a)} + \frac{1}{s-a}$$

$$= \begin{cases} 0 & \text{if } s > a \\ \infty & \text{if } s \le a \end{cases} \quad \text{(i.e, the limit doesn't exist).}$$

Thus, $\boxed{\mathcal{L}(e^{at})(s) = \frac{1}{s-a}}$ if $s > a$ (and it isn't defined otherwise).

Remark: Sometimes, the domain of F is restricted.

e.g., $f(t)$ has domain $(-\infty, \infty)$

$F(s)$ has domain (a, ∞)

(**Analogy:** e^x has domain $(-\infty, \infty)$ but its inverse function $\ln x$ has domain $(0, \infty)$).

Recall: Integration by parts (we'll need it!).

Let's rederive it: $(uv)' = u\, dv + du\, v$

$$u\, dv = (uv)' - v\, du$$

$$\boxed{\int u\, dv = uv - \int v\, du}$$

Example: Let $f(t) = t$. Compute $\mathcal{L}(F)$

$F(s) = \int_0^\infty t\, e^{-st}\, dt.$ Let $u = t$, $du = dt$; $v = -\frac{1}{s} e^{-st}$, $dv = e^{-st}\, dt$

$$\int \underbrace{t}_{u}\, \underbrace{e^{-st}\, dt}_{dv} = \underbrace{-\frac{1}{s} t e^{-st}}_{uv} + \underbrace{\frac{1}{s}\int e^{-st}\, dt}_{-\int v\, du} = -\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}$$

$$\mathcal{L}(f) = F(s) = \int_0^\infty t e^{-st}\,dt = \lim_{T\to\infty} \int_0^T t e^{-st}\,dt$$

$$= \lim_{T\to\infty} \left(\frac{-t e^{-st}}{s} - \frac{e^{-st}}{s^2} \right)\Bigg|_0^T = \lim_{T\to\infty} \left(\frac{-t e^{-st}}{s} - \frac{e^{-st}}{s^2} \right) - \left(0 - \frac{e^0}{s^2} \right) = \boxed{\frac{1}{s^2}}$$

Other common functions:

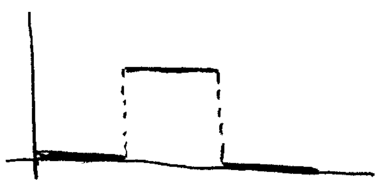
$$\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(\sin bt)(s) = \frac{b}{s^2+b^2}$$

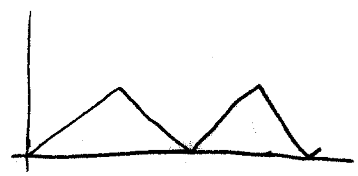
$$\mathcal{L}(\cos bt)(s) = \frac{s}{s^2+b^2}$$

We can also compute the Laplace transform of piecewise continuous & piecewise differentiable functions.

e.g.,

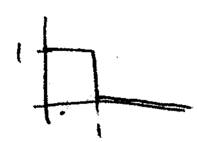


step function
(piecewise continuous)



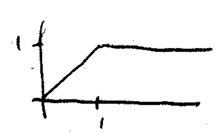
triangle wave
(piecewise differentiable)

Example: Let $f(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$



Compute $\mathcal{L}(f)(s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}\,dt = -\frac{1}{s} e^{-st}\Big|_0^1 = \boxed{\frac{-e^{-s}}{s} + \frac{1}{s}}$

Example: Let $f(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & 1 \le t < \infty \end{cases}$



We must break the integral into 2 parts:

$$\mathcal{L}(f(s)) = F(s) = \int_0^\infty e^{-st} f(t)\,dt = \underbrace{\int_0^1 t e^{-st}\,dt}_{I1} + \underbrace{\int_1^\infty e^{-st}\,dt}_{I2}$$

$$I1 = -\frac{e^{-s}}{s} - \left(\frac{e^{-s}}{s^2} - \frac{1}{s^2}\right) \qquad I2 = \lim_{T\to\infty} \left.\frac{-e^{-st}}{s}\right|_{t=1}^{T} = \frac{e^{-s}}{s}$$

$$F(s) = \left(-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} - \frac{1}{s^2}\right) + \left(\frac{e^{-s}}{s}\right) = \boxed{\frac{1}{s^2} - \frac{e^{-s}}{s^2}}$$

Properties of the Laplace transform:

- $\mathcal{L}$ is linear
- $\mathcal{L}$ "turns derivatives into multiplication"

(i) Linearity: $\mathcal{L}(a f(t) + b g(t))(s) = a\mathcal{L}(f(t))(s) + b\mathcal{L}(g(t))(s)$

i.e, you can break apart sums & pull out constants

($\mathcal{L}$ is a "linear operator")

(ii) Turns derivatives into multiplication: $\boxed{\mathcal{L}(y'(t))(s) = sY(s) - y(0)}$

Proof: $\mathcal{L}(y')(s) = \int_0^\infty y'(t)e^{-st}\,dt = \lim_{T\to\infty}\left[e^{-st}y(t) + \underbrace{s\int_0^T y(t)e^{-st}\,dt}_{s\mathcal{L}(y)(s)}\right]$

$= \lim_{T\to\infty} e^{-st}y(t)\Big|_0^T + s\mathcal{L}(y)(s)$

$= \underbrace{\lim_{T\to\infty} e^{-sT}y(T)}_{=0,\text{ as long as } |y(t)|\le Ce^{at}} - y(0) + s\mathcal{L}(y)(s) = s\mathcal{L}(y)(s) - y(0) = sY(s) - y(0).$ ✓

$=0$, as long as $|y(t)| \le Ce^{at}$. Henceforth, we will make this blanket assumption.

Similarly, $\mathcal{L}(y'')(s) = s^2Y(s) - sy(0) - y'(0)$

$\mathcal{L}(y''')(s) = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$

$\mathcal{L}(y^{(4)})(s) = s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0).$

$\vdots$

etc

Proof: $\mathcal{L}(y'') = s\mathcal{L}(y') - y(0)$

$$= s(s\mathcal{L}(y) - y(0)) - y'(0)$$

$$= s^2 Y - s\,y(0) - y'(0) \checkmark$$

The proofs of the formulas for higher derivatives is handled similarly. $\square$

More Laplace transform properties

(i) $\mathcal{L}(e^{ct} f(t))(s) = F(s-c)$

(ii) $\mathcal{L}(t\, f(t))(s) = -F'(s)$

(iii) $\mathcal{L}(t^n f(t))(s) = (-1)^n F^{(n)}(s)$

Applications/examples of this:

Example 1: $f(t) = e^{2t}\cos 3t$

Recall $\mathcal{L}(\cos 3t) = \frac{s}{s^2+9}$.

Using (i), $\mathcal{L}(e^{2t}\cos 3t) = \boxed{\frac{s-2}{(s-2)^2+9}} = F(s)$.

Example 2: $f(t) = t^2 e^{3t}$. Let $g(t) = e^{3t}$

Use (i): Recall: $\mathcal{L}(t^2) = \frac{2}{s^3} \Rightarrow$ by (i), $\mathcal{L}(t^2 e^{3t}) = \boxed{\frac{2}{(s-3)^3}}$

OR Use (iii): $\mathcal{L}(t^2 e^{3t}) = (-1)^2 F''(s) = 1\,\frac{d^2}{ds^2}\left(\frac{1}{s-3}\right) = \frac{2}{(s-3)^3}$.

Using the Laplace transform to solve ODEs

Example: Consider the IVP: $y'' - y = e^{2t}$, $y(0)=0$, $y'(0)=1$.

Apply $\mathcal{L}$: $\mathcal{L}(y'') - \mathcal{L}(y) = \mathcal{L}(e^{2t})$.

$$[s^2 Y - s\,\cancel{y(0)}^{\,0} - y'(0)] - Y = \frac{1}{s-2}$$

$$s^2 Y - 1 - Y = \frac{1}{s-2} \Rightarrow (s^2-1)Y = \frac{1}{s-2} + \frac{s-2}{s-2} = \frac{s-1}{s-2}$$

$$\Rightarrow \boxed{Y(s) = \frac{1}{(s+1)(s-2)}}$$

* The solution to the IVP is the function whose Laplace transform is

$$Y(s) = \frac{1}{(s+1)(s-2)}$$

$$\begin{tikzcd}
\begin{matrix} y''-y=e^{2t} \\ y(0)=0,\ y'(0)=1 \end{matrix} \arrow[r, "\mathcal{L}"] \arrow[d, dashed] & s^2Y-1-Y=\frac{1}{s-2} \arrow[d, "\text{solve!}"] \\
\text{solution } y(t). & Y(s)=\frac{1}{(s+1)(s-2)} \arrow[l, "\mathcal{L}^{-1}"']
\end{tikzcd}$$

To do this, write $\dfrac{1}{(s+1)(s-2)} = \dfrac{A}{(s+1)} + \dfrac{B}{(s-2)}$

Use partial fraction decomposition:

$$\frac{A}{(s+1)}\frac{(s-2)}{(s-2)} + \frac{B}{(s-2)}\frac{(s+1)}{(s+1)} = \frac{1}{(s+1)(s+2)} \Rightarrow \underbrace{(A+B)}_{=0}s + \underbrace{(B-2A)}_{=1} = 1$$

$$\Rightarrow \begin{cases} A+B=0 \\ B-2A=1 \end{cases} \Rightarrow A=-B \rightarrow 3B=1 \Rightarrow \begin{matrix} B=1/3 \\ A=-1/3 \end{matrix}$$

So, $\dfrac{1}{(s+1)(s-2)} = \dfrac{-1/3}{s+1} + \dfrac{1/3}{s-2}$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{1}{(s+1)(s-2)}\right) = \mathcal{L}^{-1}\left(\frac{-1/3}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{1/3}{s-2}\right)$$

$$= -\frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = \boxed{-\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}}$$

Example (when partial fractions fails).

Compute $\mathcal{L}^{-1}\left(\dfrac{1}{s^2+4s+13}\right)$. Problem: $s^2+4s+13$ doesn't factor.

Instead, put it in the form $\dfrac{1}{(s-a)^2+b^2}$; because $\mathcal{L}(e^{at}\sin bt) = \dfrac{b}{(s-a)^2+b^2}$.

"Complete the square!"

$$\frac{1}{(s^2+4s+4)+9} = \frac{1}{(s+2)^2+3^2} = \frac{1}{3}\frac{3}{(s+2)^2+3^2} \xmapsto{\mathcal{L}^{-1}} \boxed{\frac{1}{3}e^{-2t}\sin 3t}$$

Example: Solve the IVP: $y''-2y'-3y=0, \quad y(0)=1, \quad y'(0)=0.$

Old method: $y(t)=e^{rt}, \quad \ldots \quad e^{rt}(r^2-2r-3)=0$

$$\Rightarrow (r-3)(r+1)=0 \Rightarrow y(t)=C_1e^{3t}+C_2e^{-t}$$

Use IC's: $y(0)=C_1+C_2=1$ and $y'(t)=3C_1e^{3t}-C_2e^{-t}$

$$y'(0)=3C_1-C_2=0$$

$$\Rightarrow \begin{cases} C_1+C_2=1 \\ 3C_1-C_2=0 \end{cases} \Rightarrow C_1=\tfrac{1}{4}, \quad C_2=\tfrac{3}{4} \Rightarrow y(t)=\tfrac{1}{4}e^{3t}+\tfrac{3}{4}e^{-t}$$

New method: $y''-2y'-3y=0, \quad y(0)=1, \; y'(0)=0.$

$$\mathcal{L}(y'')-2\mathcal{L}(y')-3\mathcal{L}(y)=0$$

$$[s^2Y-s\,y(0)-y'(0)]-2[sY-y(0)]-3Y=0$$

$$[s^2Y-s-0]-2[sY-1]-3Y=0 \Rightarrow (s^2-2s-3)Y=s-2$$

$$Y=\frac{s-2}{s^2-2s-3}=\frac{A}{s-3}+\frac{B}{s+1}=\frac{s-2}{s^2-2s-3}$$

$$\frac{A}{s-3}\frac{(s+1)}{(s+1)}+\frac{B}{s+1}\frac{(s-3)}{(s-3)}=\frac{(A+B)s+(A-3B)}{(s+1)(s-3)}=\frac{s-2}{(s+1)(s-3)}$$

$$\begin{cases} A+B=1 \\ A-3B=-2 \end{cases} \Rightarrow \begin{matrix} A=\frac{1}{4} \\ B=\frac{3}{4} \end{matrix} \Rightarrow Y(s)=\frac{1/4}{s-3}+\frac{3/4}{s+1}$$

$$y(t)=\mathcal{L}^{-1}\left(\frac{1/4}{s-3}\right)+\mathcal{L}^{-1}\left(\frac{3/4}{s+1}\right)=\frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right)+\frac{3}{4}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$\Rightarrow \boxed{y(t)=\tfrac{1}{4}e^{3t}+\tfrac{3}{4}e^{-t}}$$

Summary/analysis: Consider $ay''+by'+cy=f(t), \quad y(0)=y_0, \quad y'(0)=y_1.$

$$\begin{aligned} \mathcal{L}(ay''+by'+cy) &= a\mathcal{L}(y'')+b\mathcal{L}(y')+c\mathcal{L}(y) \\ &= a(s^2Y-s\,y(0)-y'(0))+b(sY-y(0))+cY \\ &= (as^2+bs+c)Y-y_0(as+b)-ay_1=F(s). \end{aligned}$$

Thus, $$Y(s) = \underbrace{\frac{F(s)}{as^2+bs+c}}_{\text{"state-free sol'n"}} + \underbrace{\frac{(as+b)y_0 + ay_1}{as^2+bs+c}}_{\text{"input-free sol'n"}}$$

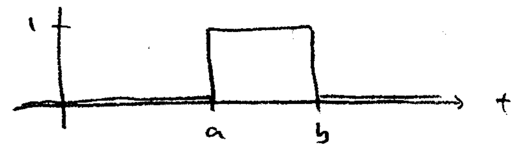
i.e., $Y(s) = Y_s(s) + Y_i(s)$, where

$Y_s(s)$ doesn't depend on the initial conditions, $y(0)=y_0$, $y'(0)=y_1$.

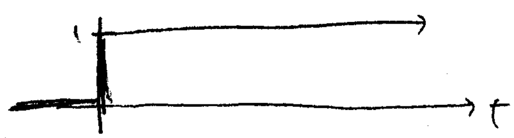
$Y_i(s)$ doesn't depend on the forcing term $f(t)$.

Discontinuous forcing terms:

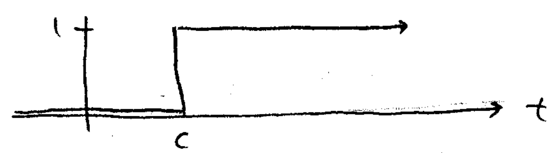
- Interval function: $H_{ab}(t) = \begin{cases} 0 & t<a \\ 1 & a \le t < b \\ 0 & b \le t < \infty \end{cases}$



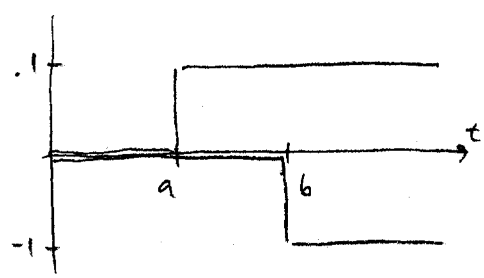
- Heavyside function: $H(t) = \begin{cases} 0 & t<0 \\ 1 & t \ge 0 \end{cases}$



- Shifted Heavyside function: $H_c(t) = \begin{cases} 0 & t<c \\ 1 & t \ge c \end{cases}$



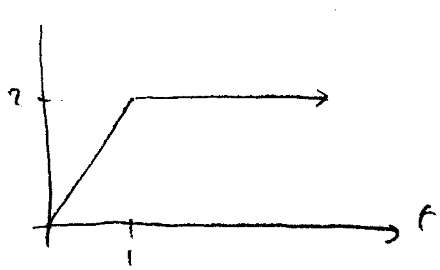
Remarks:
- $H_c(t) = H(t-c)$
- $H_{ab}(t) = H_a(t) - H_b(t) = H(t-a) - H(t-b)$



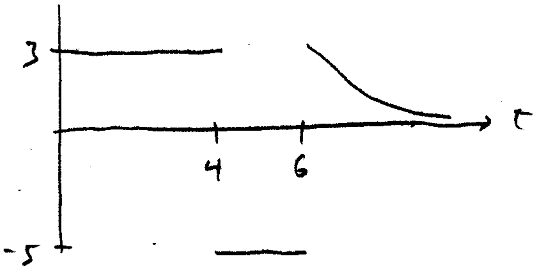
* Many piecewise continuous functions can be written using Heavyside functions

Example 1: $f(t) = \begin{cases} 2t & 0 \le t < 1 \\ 2 & t \ge 1 \end{cases}$

$$f(t) = 2t\, H_{01}(t) + 2H_1(t)$$
$$= 2t[H(t) - H(t-1)] + 2H(t-1)$$
$$= \boxed{2t\, H(t) - 2(t-1)H(t-1)}$$



Example 2: $f(t) = \begin{cases} 3 & 0 \le t < 4 \\ -5 & 4 \le t < 6 \\ e^{7-t} & 6 \le t < \infty \end{cases}$



$$f(t) = 3H_{04}(t) - 5H_{46}(t) + e^{7-t}H_6(t)$$
$$= 3[H(t) - H(t-4)] - 5[H(t-4) - H(t-6)] + e^{7-t}[H(t-6)]$$
$$= 3H(t) - 8H(t-4) + 5H(t-6) + e^{7-t}H(t-6)$$

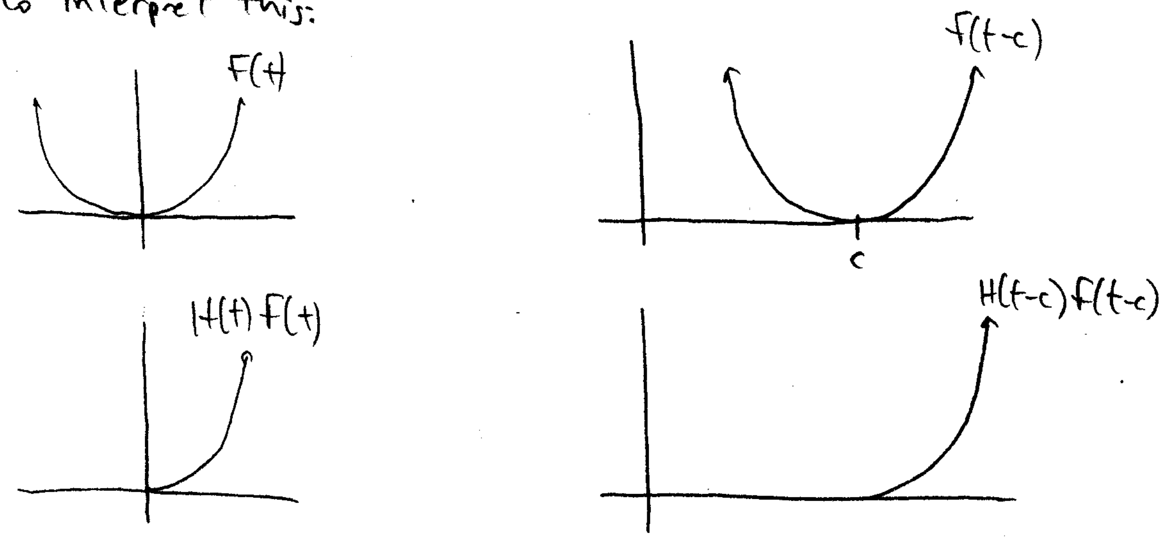
Motivation: Writing piecewise continuous functions in this form is convenient for taking their Laplace transform.

Recall: • $\mathcal{L}(H_{ab}(t)) = \int_a^b e^{-st}\,dt = \dfrac{e^{-as} - e^{-bs}}{s}$

• $\mathcal{L}(e^{ct}f(t)) = F(s-c)$

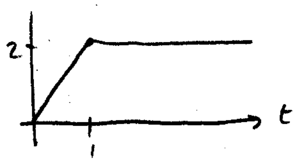
Fact: $\boxed{\mathcal{L}(H(t-c)f(t-c)) = e^{-cs}F(s)}$

How to interpret this:



Thus, $H(t-c)f(t-c)$ truncates $f(t)$ at $t=0$, then shifts it by c.
Since $f(t)$ is defined for only $t \geq 0$, we don't want $f(t-c)$ to be defined for $t < c$, so we use $H(t-c)f(t-c)$.

Example: Compute the Laplace transform of $f(t) = \begin{cases} 2t & 0 \le t < 1 \\ 2 & t \ge 1. \end{cases}$



Recall: $f(t) = 2t\,H(t) - 2(t-1)\,H(t-1)$.

$$\mathcal{L}(f(t)) = 2\mathcal{L}(t\,H(t)) - 2\mathcal{L}((t-1)H(t-1))$$

$$= \boxed{\frac{2}{s^2} - \frac{2}{s^2}e^{-s}}$$

More practice using $\mathcal{L}\{H(t-c)f(t-c)\} = e^{-cs}F(s)$:

- $\mathcal{L}\{(t-3)^2 H(t-3)\} = e^{-3s}F(s)$

  $f(t-3) = (t-3)^2$, so $f(t) = f((t+3)-3) = ((t+3)-3)^2 = t^2$

  $F(s) = \frac{1}{s^2} \Rightarrow \mathcal{L}\{(t-3)^2 H(t-3)\} = e^{-3s}F(s) = \boxed{\frac{e^{-3s}}{s^2}}$

- $\mathcal{L}\{t^2 H(t-3)\} = e^{-3s}F(s)$

  $f(t-3) = t^2$, so $f(t) = f((t+3)-3) = (t+3)^2 = t^2 + 6t + 9$

  $F(s) = \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s} \Rightarrow \mathcal{L}\{t^2 H(t-3)\} = e^{-3s}F(s) = \boxed{e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)}$

- $\mathcal{L}\{e^{t-1}H(t-1)\} = e^{-s}F(s)$

  $f(t-1) = e^{t-1}$, so $f(t) = f((t+1)-1) = e^{(t+1)-1} = e^t$

  $F(s) = \frac{1}{s-1} \Rightarrow \mathcal{L}\{e^{t-1}H(t-1)\} = \boxed{e^{-s}\frac{1}{s-1}}$

- $\mathcal{L}\{e^{7-t}H(t-6)\} = e^{-6s}F(s)$

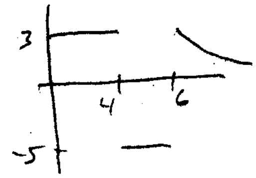
  $f(t-6) = e^{7-t}$, so $f(t) = f((t+6)-6) = e^{7-(t+6)} = e^{1-t} = e \cdot e^{-t}$

  $F(s) = e \cdot \frac{1}{s+1} \Rightarrow \mathcal{L}\{e^{7-t}H(t-6)\} = e^{-6s}F(s) = e^{-6s} e \frac{1}{s+1} = \boxed{\frac{e^{1-6s}}{s+1}}$

Trick: Given $f(t-c)$, plug in $t+c$ for t to find $f(t)$,

e.g., $f((t+c)-c) = f(t)$.

Example: Find $F(s)$, where $f(t) = \begin{cases} 3 & 0 \le t < 4 \\ -5 & 4 \le t < 6 \\ e^{7-t} & 6 \le t < \infty \end{cases}$

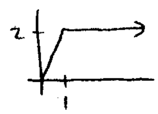


Recall: $f(t) = 3H(t) - 8H(t-4) + 5H(t-6) + e^{7-t}H(t-6)$

$$F(s) = \frac{3}{s} - \frac{8}{s}e^{-4s} + \frac{5}{s}e^{-6s} + \frac{1}{s+1}e^{1-6s} = \frac{3-8e^{-4s}+5e^{-6s}}{s} + \frac{e^{1-6s}}{s+1}.$$

Example: Solve the IVP: $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 1$,

where $f(t) = \begin{cases} 2t & 0 \le t \le 1 \\ 2 & t \ge 1 \end{cases}$



Recall: $f(t) = 2t\,H(t) - 2(t-1)H(t-1)$

$$F(s) = \frac{2}{s^2} - \frac{2}{s^2}e^{-s}.$$

Take $\mathcal{L}$ of both sides of the ODE:

$$[s^2Y - sy(0) - y'(0)] + Y = \frac{2-2e^{-s}}{s^2}$$

$$s^2Y - 1 + Y = \frac{2-2e^{-s}}{s^2}$$

$$(s^2+1)Y = \frac{2-2e^{-s}}{s^2} + 1 \Rightarrow Y(s) = \frac{2-2e^{-s}}{s^2(s^2+1)} + \frac{1}{s^2+1}$$

Partial fractions: $\dfrac{1}{s^2(s^2+1)} = \dfrac{1}{s^2} - \dfrac{1}{s^2+1}$

$$Y(s) = \frac{2}{s^2} - \frac{2}{s^2+1} - \frac{2e^{-s}}{s^2} + \frac{2e^{-s}}{s^2+1} + \frac{1}{s^2+1}$$

$$= \frac{2}{s^2} - \frac{2e^{-s}}{s^2} - \frac{1}{s^2+1} + \frac{2e^{-s}}{s^2+1}$$

$$y(t) = 2\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - 2\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) + 2\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+1}\right)$$

$$= 2t - 2(t-1)H(t-1) - \sin t + 2\sin(t-1)H(t-1)$$

$$= [2t - \sin t] + [2\sin(t-1) - 2(t-1)]H(t-1)$$

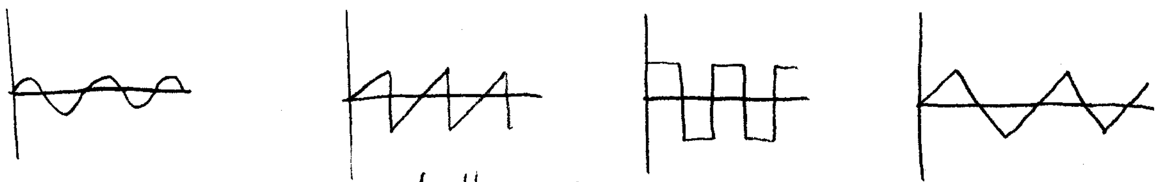
$$= \begin{cases} 2t - \sin t & 0 \le t < 1 \\ 2 + 2\sin(t-1) - \sin t & t \ge 1 \end{cases}$$

* This is the unique sol'n to the IVP $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 1$.

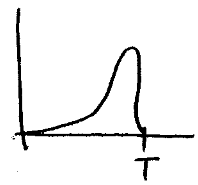
Periodic forcing terms:

- Suppose $f(t)$ is periodic. We want to compute $F(s) = \mathcal{L}(f(t))(s)$.

e.g.,

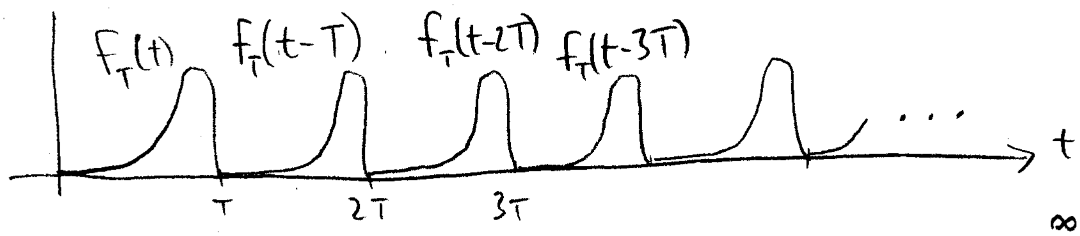


Approach: Consider the "window" $f_T(t)$, defined on $0 \le t < T$, and extended to be periodic



Then, the function $f(t) = \begin{cases} f_T(t) & 0 \le t < T \\ f_T(t-kT) & kT \le t < (k+1)T \end{cases}$

is periodic:



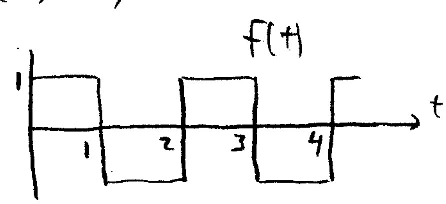
Remark: $f_T(t-kT) = f_T(t-kT)H(t-kT)$, so $f(t) = \sum_{k=0}^{\infty} f_T(t-kT)H(t-kT)$.

key point: If $|x|<1$, then $1 + x + x^2 + x^3 + \ldots = \boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$

Apply this to $F(s) = \sum_{k=0}^{\infty} \mathcal{L}\{f_T(t-kT)H(t-kT)\} = \sum_{k=0}^{\infty} e^{-kTs} F_T(s)$

$$= F_T(s) \sum_{k=0}^{\infty} e^{-kTs} = F_T(s) \sum_{k=0}^{\infty} (e^{-sT})^k = \boxed{F_T(s) \frac{1}{1-e^{-sT}}}$$

Example: Solve the IVP $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 0$, where $f(t)$ is the following square wave, of period $T = 2$.



First, compute F(s):

$$f_T(t) = H_{01}(t) - H_{12}(t) = [H(t) - H(t-1)] - [H(t-1) - H(t-2)]$$

$$= H(t) - 2H(t-1) + H(t-2)$$

$$F_T(s) = \mathcal{L}(H(t)) - 2\mathcal{L}(H(t-1)) + \mathcal{L}(H(t-2))$$

$$= \frac{1}{s} - \frac{2}{s}e^{-s} + \frac{1}{s}e^{-2s} = \boxed{\frac{(1-e^{-s})^2}{s}}$$

Now, $F(s) = F_T(s)\frac{1}{1-e^{-2s}} = \frac{F_T(s)}{(1-e^{-s})(1+e^{-s})} = \frac{(1-e^{-s})(1-e^{-s})}{s(1-e^{-s})(1+e^{-s})} = \boxed{\frac{(1-e^{-s})}{s(1+e^{-s})}}$

Back to the IVP: $\mathcal{L}(y'') + \mathcal{L}(y) = F(s)$

$$[s^2Y - sy(0) - y'(0)] + Y = \frac{1-e^{-s}}{s(1+e^{-s})}$$

$$(s^2+1)Y = \frac{1-e^{-s}}{s(1+e^{-s})} \Rightarrow \boxed{Y(s) = \frac{1-e^{-s}}{s(s^2+1)(1+e^{-s})}}$$

<u>Simplify this:</u> $Y(s) = \underbrace{\frac{1}{s(s^2+1)}}_{\frac{1}{s} - \frac{s}{s^2+1}} \cdot \underbrace{\frac{1-e^{-s}}{1+e^{-s}}}_{\text{need to further simplify.}}$

$$\frac{1-e^{-s}}{1+e^{-s}} = \frac{-(1+e^{-s})}{1+e^{-s}} + \frac{2}{1+e^{-s}} = 2\left(\frac{1}{1-(-e^{-s})}\right) = 2\sum_{n=0}^{\infty}(-1)^n e^{-ns}$$

Using $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we have $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=1}^{\infty}(-1)^n x^n$.

Thus, $Y(s) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right)\left(-1 + 2\sum_{n=0}^{\infty}(-1)^n e^{-ns}\right)$

$$= \underbrace{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)}_{\text{Call this } G(s).}\left(+1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-ns}\right) = G(s) + 2\sum_{n=1}^{\infty}(-1)^n e^{-ns}G(s).$$

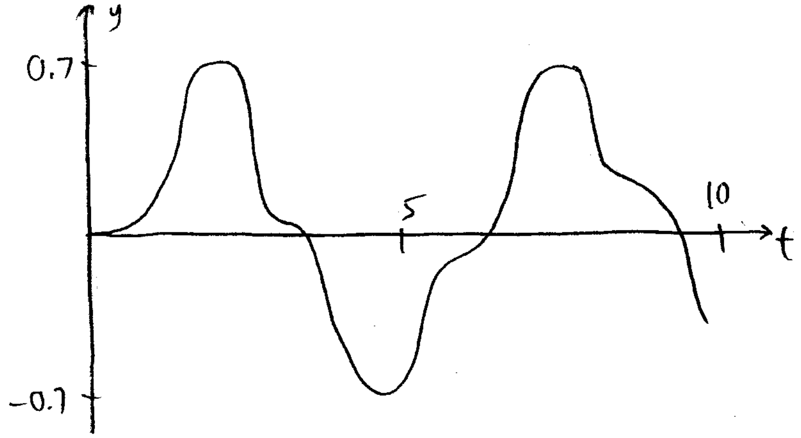
Note that $\boxed{g(t) = 1 - \cos t}$

Apply $\mathcal{L}\{g(t-n)H(t-n)\} = e^{-ns}G(s)$.

$$Y(s) = G(s) + 2\sum_{n=1}^{\infty}(-1)^n e^{-ns}G(s) \Longrightarrow y(t) = g(t) + 2\sum_{n=1}^{\infty}(-1)^n g(t-n)H(t-n)$$

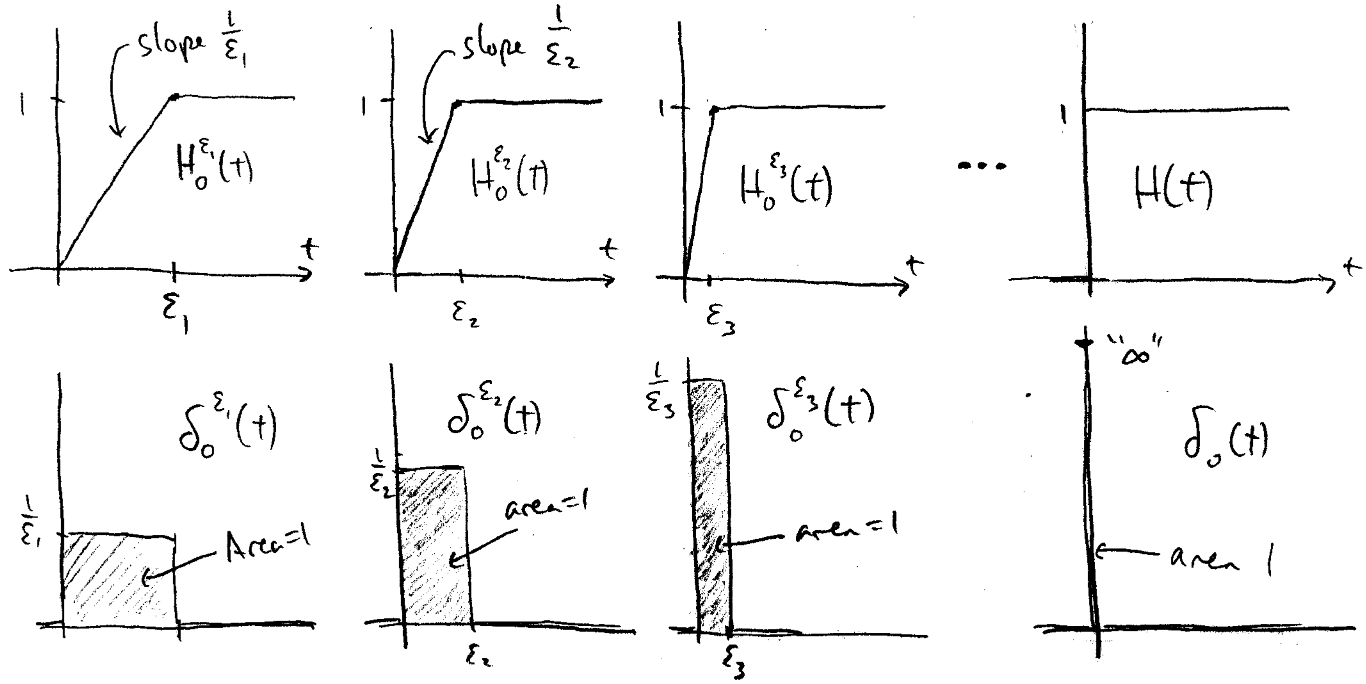
$$\boxed{y(t) = (1-\cos t)H(t) + 2\sum_{n=1}^{\infty}(-1)^n\left[1-\cos(t-n)\right]H(t-n)}$$

This is a superposition of infinitely many waves:



Question: What is the "derivative" of the Heavyside function?

Technically, it's not defined, but what "should" it be?



Def: The delta function is $\delta_p(t) = \lim_{\varepsilon \to 0} \delta_p^{\varepsilon}(t)$; $\quad \delta_p(t) = \begin{cases} 0 & t \neq p \\ \infty & t = p \end{cases}$

Technically, it's not really a function, but it's useful!

It has infinite height, infintesimal with, and integral 1.

Properties of the Delta function:

- $\int_{-\infty}^{\infty} \delta_p(t)\, dt = 1 \quad \Rightarrow \quad \int_{-\infty}^{\infty} \delta_p(t) f(t)\, dt = f(p)$
- $\mathcal{L}(\delta_0) = 1 \quad \Rightarrow \quad \mathcal{L}(\delta_p) = e^{-sp}$

So now we can take the inverse Laplace transform of a constant or an exponential.

* The Delta function models a <u>unit impulse force</u> (finite force over an infintesimal time interval (or realistically, a very small interval). e.g., Exerting a force by hitting something with a hammer.

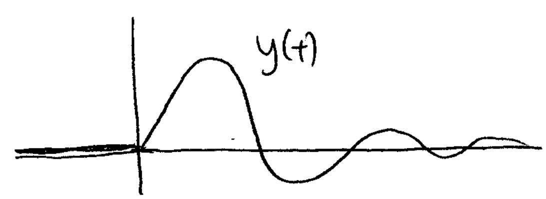
<u>Example:</u> Solve the IVP $y'' + 2y' + 2y = \delta_0(t)$, $y(0) = 0$, $y'(0) = 0$

First, take the Laplace transform: $\mathcal{L}(y'' + 2y' + 2y) = \mathcal{L}(\delta_0(t))$

$$(s^2 + 2s + 2)Y = 1$$

$$\Rightarrow Y(s) = \frac{1}{s^2 + 2s + 2} = \frac{1}{(s+1)^2 + 1} \quad \Rightarrow \quad \boxed{y(t) = e^{-t} \sin t}$$

<u>Remark:</u> $y(0) = 0$ but $y'(0) = 1$. (This doesn't match the initial conditions!)



This is okay, because $y(t)$ is only defined for $t > 0$, so what we really have is $H(t)\, y(t) = \begin{cases} 0 & t \le 0 \\ e^{-t} \sin t & t > 0 \end{cases}$

This isn't even differentiable at $t = 0$, so technically, the derivative isn't defined. But $\lim_{t \to 0^-} \frac{d}{dt}\left(H(t)\, y(t)\right) = 0$, and that's "good enough."