1. The joint density of X and Y is given by

$$f_{X,Y}(x,y) = e^{-(x+y)} \qquad 0 \le x < \infty, \quad 0 \le y < \infty\,.$$

   (a) Find $P(X < Y)$.

   (b) Find $P(X < a)$.

   (c) Compute the marginal densities $f_X(x)$ and $f_Y(y)$.

   (d) Are X and Y independent?

2. Consider the joint pdf

$$f_{X,Y}(x,y) = \frac{1}{x} \qquad 0 < y < x < 1.$$

   (a) Compute the marginal densities $f_X(x)$ and $f_Y(y)$.

   (b) Compute the expected values $E[X]$ and $E[Y]$.

3. Let X and Y be independent. For each of the following situations, write down the joint density function for X and Y.

   (a) $X \sim \text{Unif}[0,1]$ and $Y \sim \text{Unif}[0,2]$.

   (b) $X \sim \text{Normal}(2,4)$ and $Y \sim \text{Exp}(\lambda = 5)$.

   (c) $X, Y \sim \text{Exp}(\lambda = 2)$.

4. Let (X,Y) have joint probability mass function (pmf)

$$p_{X,Y}(x,y) = \frac{x+y}{21} \qquad x = 1,2,3; \quad y = 1,2.$$

   (a) Find $p_X(x)$ and $p_Y(y)$, the marginal pmfs of X and Y.

   (c) Are X and Y independent?

   (d) Find $P(X > Y)$.

   (e) Find $P(Y = 2X)$.

   (f) Find $P(X + Y = 3)$.

   (g) Find $P(Y = 3 \mid X = 2)$

5. Let (X,Y) have joint probability density function (pdf)

$$f_{X,Y}(x,y) = k \qquad 0 \le x \le 2, \quad 0 \le y \le 1, \quad \text{and} \quad 2y \le x.$$

   (a) Find k.

   (b) Find $P(X \ge 3Y)$.

   (c) Compute the marginal densities $f_X(x)$ and $f_Y(y)$.

   (d) Are X and Y independent?

6. Let

$$f_{X,Y}(x,y) = 2 \qquad 0 \le x \le 1, \quad 0 \le y \le 1, \quad 0 \le x + y \le 1.$$

(a) Find the conditional pdf for $Y|x$.

(b) Find $\mathrm{E}(Y \mid X = x)$.

(c) Find the conditional pdf for $X|y$.

(d) Find $\mathrm{E}(X \mid Y = y)$.

7. Let X and Y have joint pmf

$$p_{X,Y}(x,y) = \frac{x+y}{32} \qquad x = 1,2; \quad y = 1,2,3,4$$

Find the conditional pmf of Y given $X = x$. Use this to find $P(Y = 2 \mid |X = 1)$.

8. If $Z \sim \text{Normal}(0,1)$, then it is well-known that $\mathrm{E}(Z) = 0$, $\mathrm{E}(Z^2) = 1$, and $\mathrm{E}(Z^3) = 3$. Let X be independent of Z with $P(X = 1) = P(X = -1) = 0.5$. Finally let $Y = XZ$.

(a) Show $\mathrm{E}(X) = 0$ and $\mathrm{E}(Y) = 0$.

(b) Show $\mathrm{E}(YZ) = 0$ and that $\mathrm{Cov}(Y, Z) = 0$.

(c) Argue that Y and Z are highly dependent. Hint: Find $P(|Z| > 1 \,\big|\, |Y| < 1)$.

(d) Conclude that $\mathrm{Cov} = 0$ *does not imply* independence.

9. Let the joint density of X and Y be

$$f_{X,Y}(x,y) = x + y \qquad 0 < x < 1 \quad 0 < y < 1.$$

(a) Find the marginal distributions of X and Y.

(b) Find $E(X)$ and $E(XY)$.

(c) $\mathrm{Cov}(X,Y)$. Are X and Y independent?

10. A manufacturing system depends on a part, but this part has a back-up. Let X be the time until failure of the first part, and Y the time until failure of the back-up. Assume that X and Y are jointly continuous, with joint density

$$f_{X,Y}(x,y) = e^{-y} \qquad 0 < x < y < \infty.$$

(a) Compute the marginal densities $f_X(x)$ and $f_Y(y)$.

(b) Find the conditional density for Y given $X = x$.

(c) Compute the expected values $\mathrm{E}[X]$ and $\mathrm{E}[Y]$.

(d) Compute the variances $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$.

(e) Compute the covariance $\mathrm{Cov}(X,Y)$ and correlation $\rho(X,Y)$ of X and Y.