TOPICS: PDES ON UNBOUNDED DOMAINS. LAPLACE & FOURIER TRANSFORMS.

1. Recall that the solution to the initial value problem for the heat equation, where $x \in \mathbb{R}$ and $t > 0$:

$$u_t = c^2 u_{xx}, \qquad u(x,0) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases} \qquad \text{is} \quad u(x,t) = \int_{-\infty}^{\infty} h(x) \frac{1}{\sqrt{4\pi c^2 t}} e^{-(x-y)^2/4c^2 t}\, dy,$$

where $h(x) = u(x,0)$. Sketch the heat distribution at $t = 0$ and several larger values of t on the same set of axes. Then change variables by setting $z = (y-x)/\sqrt{4c^2 t}$ and write the above solution in terms of the Gauss error function, $\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-r^2}\, dr$.

2. Starting with *d'Alembert's formula*, the general solution to the wave equation on $x \in \mathbb{R}$, solve the following initial value problem:

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = \frac{1}{1 + x^2/4}, \qquad u_t(x,0) = 0.$$

Then repeat for the following IVP:

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = 0, \qquad u_t(x,0) = \frac{1}{1 + x^2/4}.$$

Describe what each IVP models. Compare and contrast the differences in the solutions. For each one, sketch the function $u(x,t)$ for $t = 0$, and several larger values of t on the same set of axes.

3. Solve the I/BVP for the heat equation on the semi-infinite domain $x > 0$, $t > 0$:

$$u_t = c^2 u_{xx}, \qquad u(0,t) = 0, \qquad u(x,0) = 1.$$

Write your answer in terms of the erf function. Sketch a graph of $u(x,t)$ at $t = 0$ and several larger values of t. What is the long-term behavior?

4. The following I/BVP models what happens to a falling cable that is lying on a table that is suddenly removed:

$$u_{tt} = c^2 u_{xx} - g, \qquad u(0,t) = 0, \qquad u(x,0) = u_t(x,0) = 0.$$

Assume that the domain of $u(x,t)$ is $x > 0$ and $t > 0$. Solve this PDE using Laplace transforms and draw several time snapshots of the solution.

5. Use a Fourier transform to solve the following initial value problem for the inhomogeneous heat equation, where $x \in \mathbb{R}$ and $t > 0$:

$$u_t = u_{xx} + h(x,t), \qquad u(x,0) = 0.$$

6. Use a Fourier transform to solve the following initial value problem for the free Schrödinger equation, where $x \in \mathbb{R}$ and $t > 0$:

$$u_t = i u_{xx}, \qquad u(x,0) = e^{-x^2}.$$