Lecture 3.7: Fourier transforms

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What is a Fourier transform?

Definition

Suppose $f\colon \mathbb{R} \to \mathbb{C}$ vanishes outside some finite interval. Its Fourier transform is defined by

$$\mathcal{F}(f) = \widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = \lim_{L\to\infty}\int_{-\pi L}^{\pi L} f(x)e^{-i\omega x}\,dx.$$

Suppose f vanishes outside $[-\pi L, \pi L]$. Extend this function to be $2\pi L$-periodic. Note that

$$\frac{1}{2\pi L}\widehat{f}(n/L) = \frac{1}{2\pi L}\int_{-\infty}^{\infty} f(x)e^{-inx/L}\,dx = \frac{1}{2\pi L}\int_{-\pi L}^{\pi L} f(x)e^{-inx/L}\,dx = c_n.$$

Thus, we can write $f(x)$ as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\pi \frac{n}{L}x} = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi L}\widehat{f}\left(\tfrac{n}{L}\right)e^{i\pi \frac{n}{L}x}.$$

Let $\omega_n = \frac{n}{L}$ and $\Delta\omega = \frac{1}{L}$. Taking the limit as $\Delta\omega \to 0$ yields

$$f(x) = \lim_{L\to\infty}\sum_{n=-\infty}^{\infty} c_n e^{i\pi \frac{n}{L}x} = \frac{1}{2\pi}\lim_{\Delta\omega\to 0}\sum_{n=-\infty}^{\infty} \widehat{f}(\omega_n)e^{i\pi\omega_n x}\Delta\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(\omega)e^{i\pi\omega x}d\omega.$$

This is called the inverse Fourier transform of $\widehat{f}(\omega)$, also denoted $\mathcal{F}^{-1}(\widehat{f})$.

Example: a rectangular pulse

Consider a $2L$-periodic function defined by $f(x) = \begin{cases} 1 & -0.5 < x < 0.5 \\ 0.5 & x = 0.5 \\ 0 & 0.5 < |x| < L. \end{cases}$

- If $L = 1$, compute its complex Fourier series.
- How does this compare to $L = 2$? To $L = 200$?
- What is its Fourier transform?

A "continuous" version of a Fourier series

Every continuous function $f\colon[-\pi,\pi]\to\mathbb{C}$ can be decomposed into a discrete sum of complex exponentials:

$$f(x)=\sum_{n=-\infty}^{\infty}c_ne^{inx},\qquad c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}\,dx,\qquad \textit{let } \omega=1.$$

Every continuous function $f\colon[-2\pi,2\pi]\to\mathbb{C}$ can be decomposed into a discrete sum of complex exponentials:

$$f(x)=\sum_{n=-\infty}^{\infty}c_ne^{inx},\qquad c_n=\frac{1}{4\pi}\int_{-2\pi}^{2\pi}f(x)e^{-inx/2}\,dx,\qquad \textit{let } \omega=1/2.$$

Every continuous function $f\colon[-200\pi,200\pi]\to\mathbb{C}$ can be decomposed into a discrete sum of complex exponentials:

$$f(x)=\sum_{n=-\infty}^{\infty}c_ne^{inx},\qquad c_n=\frac{1}{400\pi}\int_{-200\pi}^{200\pi}f(x)e^{-inx/200}\,dx,\qquad \textit{let } \omega=1/200.$$

Now take the limit as $L\to\infty\dots$

Every continuous function $f\colon(-\infty,\infty)\to\mathbb{C}$ can be decomposed into a ~~discrete sum~~ integral of complex exponentials:

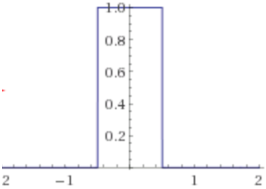
$$f(x)=\int_{-\infty}^{\infty}c_\omega e^{i\omega x}d\omega,\qquad c_\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}\,dx=\frac{1}{2\pi}\widehat{f}(\omega).$$

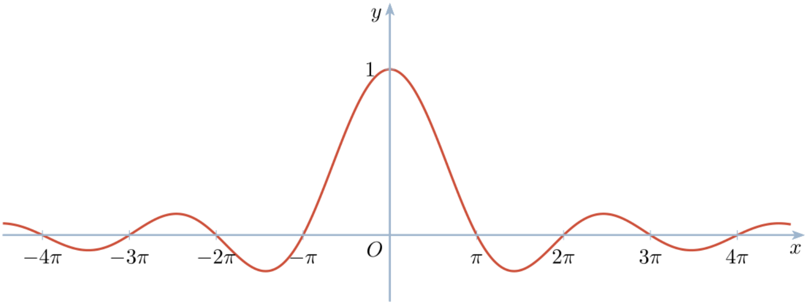
The sine cardinal (sinc) function

The Fourier transform of the "rectangle function" in the previous example is

$$\text{sinc}(x) = \begin{cases} 1 & x = 0 \\ \frac{\sin x}{x} & x \neq 0 \end{cases}$$

This is called the "sampling function" in signal processing.





"Evil twins" of the Fourier transform

- Our *Fourier transform* and inverse transform:

$$\widehat{f}(\omega) := \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx, \qquad \text{and} \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(x)e^{i\omega x}\,d\omega$$

- The *opposite Fourier transform* and its inverse:

$$\check{f}(\omega) := \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx, \qquad \text{and} \qquad f(x) = \int_{-\infty}^{\infty} \check{f}(x)e^{i\omega x}\,d\omega$$

- The *symmetric Fourier transform* and its inverse:

$$\widehat{\mathrm{f}}(\omega) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx, \qquad \text{and} \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{\mathrm{f}}(x)e^{i\omega x}\,d\omega$$

- The *canonical Fourier transform* and its inverse:

$$\widetilde{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}\,dx, \qquad \text{and} \qquad f(x) = \int_{-\infty}^{\infty} \widetilde{f}(x)e^{2\pi i\xi x}\,d\xi$$

This last definition is motivated by of the relation $\omega = 2\pi\xi$ between angular frequency ω (radians per second) and oscillation frequency ξ (cycles per second, or "Hertz").

It is easy to go between these definitions:

$$\widehat{f}(\omega) = 2\pi\check{f}(\omega) = \sqrt{2\pi}\,\widehat{\mathrm{f}}(\omega) = \widetilde{f}\left(\tfrac{\omega}{2\pi}\right) = \widetilde{f}(\xi).$$

Recall that the Laplace transform of a function $f(t)$ is

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}.$$

To get its Fourier transform, just plug in $s = i\omega$:

$$F(i\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} = F(s)\Big|_{s=i\omega}.$$

Because of this, these transforms share many similar properties:

Property	time-domain	frequency domain
Linearity	$c_1 f_1(t) + c_2 f_2(t)$	$c_1 \widehat{f_1}(\omega) + c_2 \widehat{f_2}(\omega)$
Time / phase-shift	$f(t - t_0)$	$e^{-i\omega t_0}\widehat{f}(\omega)$
Multiplication by exponential	$e^{i\nu t} f(t)$	$\widehat{f}(\omega - \nu)$
Dilation by $c > 0$	$f(ct)$	$\frac{1}{c}\widehat{f}(\omega/c)$
Differentiation	$\dfrac{df(t)}{dt}$	$i\omega \widehat{f}(\omega)$
Multiplication by t	$tf(t)$	$-\dfrac{d}{d\omega}\hat{f}(\omega)$
Convolution	$f_1(t) * f_2(t)$	$\widehat{f_2}(\omega) \cdot \widehat{f_2}(\omega) = (\widehat{f_1} * \widehat{f_2})(\omega)$