Lecture 3.8: Pythagoras, Parseval, and Plancherel

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Math 4340, Advanced Engineering Mathematics

Our journey from $\mathbb{R}^n$ to Fourier transforms

In the beginning of this class, we started with standard Euclidean space, $\mathbb{R}^n$. The dot product gave us a notion of geometry: lengths, angles, and projections. Our favorite orthonormal basis was $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$.

Moving on to the space $\text{Per}_{2L}(\mathbb{C})$ of piecewise $2L$-periodic functions, we defined an inner product, which gave us a notion of geometry: norms, angles, and projections. Our favorite orthonormal basis was $\left\{e^{i\pi nx/L} \mid n \in \mathbb{Z}\right\}$.

Let $L^2(\mathbb{R})$ be the set of square-integrable functions, i.e., $||f||^2 := \int_{\mathbb{R}} |f|^2 dx < \infty$. The Fourier transform of $f \in L^2(\mathbb{R})$ can be thought of as a "continuous" version of a Fourier series.

Definition

We defined the Fourier transform of $f \in L^2(\mathbb{R})$ and its inverse transform as

$$\widehat{f}(\omega) := \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx, \qquad \text{and} \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(x)e^{i\omega x}d\omega.$$

Think of ω as angular frequency. Another definition of the Fourier transform was in terms of oscillatory frequency, $\xi = \omega/(2\pi)$:

$$\widetilde{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}dx, \qquad \text{and} \qquad f(x) = \int_{-\infty}^{\infty} \widetilde{f}(x)e^{2\pi i\xi x}d\xi$$

Generalizations of a celebrated theorem of the ancient Greeks

Pythagorean theorem for vectors in $\mathbb{R}^n$

Given a vector $\mathbf{v} = c_1\mathbf{e}_1 + \cdots + c_n\mathbf{e}_n$,

$$||\mathbf{v}||^2 = \langle \mathbf{v}, \mathbf{v} \rangle = \sum_{i=1}^{n} c_i^2.$$

Parseval's identity for Fourier series in $\text{Per}_{2L}(\mathbb{C})$

Given a Fourier series $f(x) = \displaystyle\sum_{n=-\infty}^{\infty} c_n e^{i\pi nx/L}$,

$$||f||^2 = \langle f, f \rangle = \frac{1}{2L}\int_{-L}^{L} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} c_n^2.$$

Plancherel's theorem for Fourier transforms in $L^2(\mathbb{R})$

If f is square-integrable, then

$$||\widehat{f}||^2 = \langle \widehat{f}, \widehat{f} \rangle = \int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2 d\xi = \int_{-\infty}^{\infty} |f(x)|^2 dx = \langle f, f \rangle = ||f||^2.$$

Parseval's identity for Fourier transforms

Plancherel's theorem says that the Fourier transform is an isometry. It follows from a more general result.

Parseval's identity for Fourier transforms

If $f, g \in L^2(\mathbb{R})$, then $\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle$.

Proof

Introduction

Parseval's identity for real Fourier series

If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$, then

$$||f||^2 = \langle f, f \rangle := \frac{1}{L} \int_{-L}^{L} \big(f(x)\big)^2 dx = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 .$$

An application of Parseval's identity

Sum of inverse squares

Compute the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$.