Lecture 4.2: Symmetric and Hermitian matrices

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Math 4340, Advanced Engineering Mathematics

Motivation

Recall the following concept from linear algebra.

Definition

Let **A** be an $n \times n$ matrix and $\mathbf{v} \in \mathbb{R}^n$ be a vector. If $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{C}$, then $\mathbf{v}$ is an eigenvector with eigenvalue λ.

Remark

The eigenvalues λ_1, λ_2 of a 2×2 matrix **A** are the roots of a degree-2 polynomial. There are 3 cases:

(i) distinct, real roots: $-\infty < \lambda_1 < \lambda_2 < \infty$,

(ii) complex roots: $\lambda_{1,2} = a \pm bi$,

(iii) repeated roots: $\lambda_1 = \lambda_2$.

Symmetric matrices

Theorem

If a (real-valued) matrix $\mathbf{A}$ is symmetric, i.e., $\mathbf{A}^T = \mathbf{A}$, then:

1. All eigenvalues are real.
2. There is a full orthonormal set (a basis!) of eigenvectors.

Example

Compute the eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Symmetric matrices

Theorem

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Non-examples

Compute the eigenvalues and eigenvectors of:

- $\mathbf{B} = \begin{bmatrix} 3 & -9 \\ 4 & -3 \end{bmatrix}$
- $\mathbf{C} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Hermitian matrices

Theorem

If a (complex-valued) matrix $\mathbf{A}$ is Hermitian, i.e., $\mathbf{A}^T = \overline{\mathbf{A}}$ then

1. All eigenvalues are real.
2. There is a full orthonormal set (a basis!) of eigenvectors.

Example

Compute the eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

Hermitian matrices

Theorem

If a (complex-valued) matrix $\mathbf{A}$ is Hermitian, i.e., $\mathbf{A}^T = \overline{\mathbf{A}}$ then

1. All eigenvalues are real.
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Non-example

Compute the eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

Self-adjoint mappings

Definition

Let V be a vector space with inner product $\langle -, - \rangle$. A linear map $\mathbf{A}\colon V \to V$ is self-adjoint if

$$\langle \mathbf{A}\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{A}\mathbf{w} \rangle, \qquad \text{for all } \mathbf{v}, \mathbf{w} \in V.$$

Remarks

- Using the standard dot product in $V = \mathbb{R}^n$, a matrix $\mathbf{A}$ is self-adjoint iff it is symmetric.
- Using the standard inner product in $V = \mathbb{C}^n$, a matrix $\mathbf{A}$ is self-adjoint iff it is Hermitian.

Theorem (proof in the next lecture)

If $\mathbf{A}$ is self-adjoint, then:

1. All eigenvalues are real.
2. Eigenvectors corresponding to distinct eigenvalues are orthogonal.

Think about what this means in (infinite-dimensional) vector spaces of functions, where differential operators are linear maps.