Lecture 7.1: Harmonic functions and Laplace's equation

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Higher dimensional PDEs

Recall the del operator ∇ from vector calculus:

$$\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right), \qquad \Delta := \nabla \cdot \nabla = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

In $\mathbb{R}^n$

- Heat equation: $u_t = c^2 \Delta u$
- Wave equation: $u_{tt} = c^2 \Delta u$

Remark

Steady state solutions:

- occur for the heat equation (*heat dissipates*)
- do not occur for the wave equation (*waves propagate*)

Definition

A steady-state solution means $u_t = 0$. Thus, all steady-state solutions satisfy $u_t = c^2 \Delta u = 0$, i.e.,

$$\Delta u = 0 \qquad \Longrightarrow \qquad \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$

A function u is harmonic if $\Delta u = 0$.

Properties of harmonic functions

Key properties

- The graphs of harmonic functions ($\Delta f = 0$) are as flat as possible.
- If f is harmonic, then for any closed bounded region R, the function f achieves its minimum and maximum values on the boundary, ∂R.

Examples of harmonic functions

Solving Laplace's equation on a bounded domain

Example 1a

Solve the following BVP for Laplace's equation:

$$u_{xx} + u_{yy} = 0, \qquad u(0, y) = u(x, 0) = u(\pi, y) = 0, \quad u(x, \pi) = x(\pi - x)\,.$$

Solving Laplace's equation on a bounded domain

Example 1b

Solve the following BVP for Laplace's equation:

$$u_{xx} + u_{yy} = 0, \qquad u(0,y) = u(x,0) = u(x,\pi) = 0, \quad u(\pi,y) = y(\pi - y)\,.$$

Solving Laplace's equation on a bounded domain

Example 1c

Solve the following BVP for Laplace's equation:

$$u_{xx} + u_{yy} = 0, \qquad u(0,y) = u(x,0) = 0, \quad u(x,\pi) = x(\pi - x), \quad u(\pi,y) = y(\pi - y)\,.$$

Unbounded domains and Fourier transforms

Example 2

Solve the following BVP for Laplace's equation, where $x \in \mathbb{R}$ and $y > 0$, and the solution u is bounded as $y \to \infty$:

$$u_{xx} + u_{yy} = 0, \qquad u(x,0) = f(x).$$