Lecture 7.5: Three PDEs on a disk

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Overview

Last time

In polar coordinates, the Laplacian operator is

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} = \frac{1}{r}\partial_r + \partial_r^2 + \frac{1}{r^2}\partial_\theta^2.$$

Its eigenvalues and eigenfunctions (if $0 \leq r \leq 1$ and $0 \leq \theta < 2\pi$) are

$$\lambda_{nm} = \omega_{nm}^2, \qquad f_{nm}(r,\theta) = \cos(n\theta)\, J_n(\omega_{nm} r), \qquad g_{nm}(r,\theta) = \sin(n\theta)\, J_n(\omega_{nm} r),$$

where ω_{nm} is the m^{th} positive root of $J_n(r)$, the Bessel function of the first kind of order n.

This time

In this lecture, we'll solve boundary value problems for

- Laplace's equation (inhomogeneous BCs)
- the heat equation
- the wave equation

These arise naturally when solving the heat equation over a circular plate or the wave equation over a circular drum.

Laplace's equation in a disk, inhomogeneous BCs

Example 1

Solve the following BVP for Laplace's equation in polar coordinates

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \qquad u(1,\theta) = h(\theta), \qquad u(r,\theta+2\pi) = u(r,\theta).$$

Diffusion on a disk

Example 2

Solve the following B/IVP for the heat equation in polar coordinates

$$u_t = c^2 \Delta u, \qquad \underbrace{u(1,\theta,t)=0}_{\text{Dirichlet BC}}, \quad \underbrace{u(r,\theta+2\pi,t)=u(r,\theta,t)}_{\text{periodic BC}}, \qquad \underbrace{u(r,\theta,0)=h(r,\theta)}_{\text{initial condition}}.$$

The wave equation on a drum

Example 3

Solve the wave equation $u_t = c^2 \Delta u$ subject to the following BCs and ICs:

$$\underbrace{u(1,\theta,t)=0}_{\text{Dirichlet BC}}, \quad \underbrace{u(r,\theta+2\pi,t)=u(r,\theta,t)}_{\text{periodic BC}}, \quad \underbrace{u(r,\theta,0)=h_1(r,\theta), \quad u_t(r,\theta,0)=h_2(r,\theta)}_{\text{initial conditions}}.$$

Eigenfunctions of the Laplacian in the unit disk

$$\lambda_{nm} = \omega_{nm}^2, \qquad f_{nm}(r,\theta) = \cos(n\theta)\, J_n(\omega_{nm} r), \qquad g_{nm}(r,\theta) = \sin(n\theta)\, J_n(\omega_{nm} r)$$