MTHSC 851/852 (Abstract Algebra)
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HW 13
Due Monday, September 21, 2009

(1) Show that $[\mathbb{A} : \mathbb{Q}]$ is infinite.
(2) Let F be a field extension of $\mathbb{Q}$ such that $[F : \mathbb{Q}] = 2$. Prove that there is a unique square-free integer m such that $F = \mathbb{Q}(\sqrt{m})$.
(3) Let $S = \{\sqrt{m} \mid m \in \mathbb{N} \text{ is prime}\}$.
  (a) Show that $\mathbb{Q}[S] = \mathbb{Q}(S)$.
  (b) Prove that for $n \in \mathbb{N} \cup \{0\}$ and any choice of $n+1$ distinct elements $a_1, \ldots, a_{n+1} \in S$, $a_{n+1} \notin \mathbb{Q}(a_1, \ldots, a_n)$.
  (c) Deduce that for any finite subset $T \subseteq S$ we have $[\mathbb{Q}(T) : \mathbb{Q}] = 2^{|T|}$. Use this fact to argue that $[\mathbb{Q}(S) : \mathbb{Q}]$ is infinite.
  (d) Deduce further that $\mathbb{Q}(T) \neq \mathbb{Q}(U)$, whenever T and U are distinct subsets of S.
(4) Let $F \subseteq \mathbb{C}$ be a splitting field for $x^3 - 2$ over $\mathbb{Q}$.
  (a) Describe F, i.e. give some generators for a field extension of $\mathbb{Q}$.
  (b) Compute the dimension of F over $\mathbb{Q}$. Justify your claims.
  (c) Is F isomorphic to a subfield of $\mathbb{R}$? Prove or disprove.
(5) Let $a_1, a_2 \in \mathbb{C}$ be any two numbers which are algebraic over $\mathbb{Q}$. Show that if there exists an isomorphism $\phi : \mathbb{Q}(a_1) \to \mathbb{Q}(a_2)$, leaving $\mathbb{Q}$ elementwise fixed and satisfying $\phi(a_1) = a_2$, then a_1 and a_2 have the same minimal polynomial over $\mathbb{Q}$.
(6) Fix a field F, and define a category $\mathfrak{C}_F$ whose objects are the extension fields of F. Define morphisms in $\mathfrak{C}_F$ in such a fashion that the algebraic closure $\bar{F}$ of F arises as the solution of a universal mapping problem. Carefully formulate this problem and prove all of your claims.