MTHSC 851/852 (Abstract Algebra)
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HW 18
Due Monday, December 7th, 2009

(1) (a) Suppose M is an R-module, that $x, y \in \mathrm{Tor}(M)$ with $|x| = r$, $|y| = s$, and that r and s are relatively prime in R. Show that $|x + y| = rs$.
 (b) Give an example of an R-module M over a commutative ring R where $\mathrm{Tor}(M)$ is not a submodule.

(2) Suppose R is a commutative ring and M is an R-module. A submodule N is called *pure* if $rN = rM \cap N$ for all $r \in R$.
 (a) Show that any direct summand of M is pure.
 (b) If M is torsion free and N is a pure submodule show that M/N is torsion free.
 (c) If M/N is torsion free show that N is pure.

(3) If R is a commutative ring with 1 and M is an R-module define a function $\phi : x \to \hat{x}$ from M to its double dual $M^{**} = (M^*)^*$ by setting $\hat{x}(f) = f(x)$, all $f \in M^*$. Show that ϕ is an R-homomorphism. Under what circumstances is ϕ a monomorphism?

(4) If R is a commutative ring with 1 and x_1, x_2 are distinct indeterminantes show that $R[x_1, x_2]$ and $R[x_1] \otimes_R R[x_2]$ are isomorphic as R-algebras.

(5) Suppose A is a finitely generated abelian group.
 (a) Compute $A \otimes_Z \mathbb{Q}$.
 (b) Define $f : A \to A \otimes_Z \mathbb{Q}$ by setting $f(a) = a \otimes 1$ for all $a \in A$. Show that f is a homomorphism. Under what circumstances if f a monomorphism?

(6) If A is an abelian group show that $\mathbb{Z}_n \otimes_{\mathbb{Z}} A \cong A/nA$.

(7) If $K \to M \to N \to 0$ is an exact sequence of left R-modules and L is a right R-module show that $L \otimes_R K \to L \otimes_R M \to L \otimes_R N \to 0$ is an exact sequence of abelian groups.