MTHSC 851 (Abstract Algebra)
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HW 2
Due wednesday Jan. 28, 2009

(1) Suppose G is finite, p is the smallest prime dividing $|G|$, $H \leq G$, and $[G : H] = p$. Show that $H \lhd G$.
(2) Suppose $[G : H]$ is finite. Show that there is a normal subgroup K of G with $K \leq H$, such that $[G : K]$ is finite.
(3) Suppose $H \leq S_n$ but $H \not\leq A_n$. Show that $[H : A_n \cap H] = 2$.
(4) Prove that if H and K are normal subgroups of a group G and $HK = G$ then

$$G/(H \cap K) \cong (G/H) \times (G/K).$$

(5) Prove the *tower law*: If $K \leq H \leq G$, then $[G : K] = [G : H][H : K]$.
(6) Suppose G is finite, $H \leq G$, $[G : H] = n$, and $|G| \not| n!$. Show that there is a normal subgroup K of G, $K \neq 1$, such that $K \leq H$.
(7) If $|G| = p^n$ for some prime p, and $1 \neq H \lhd G$, show that $H \cap Z(G) \neq 1$.
(8) If $A, B \leq G$ and $y \in G$ define (A, B)-*double coset* $AyB = \{ayb \mid a \in A, b \in B\}$. Show that G is the disjoint union of its (A, B)-double cosets. Show that $|AyB| = [A^y : A^y \cap B] \cdot |B|$ if A and B are finite.
(9) Let G be a group of order 15, which acts on a set S with 7 elements. Show the group action has a fixed point.
(10) Suppose G acts on S, $x \in G$, and $x \in S$. Show that $\mathrm{Stab}_G(xs) = x \,\mathrm{Stab}_G(s) x^{-1}$.
(11) Prove that if G contains no subgroup of index 2, then any subgroup of index 3 is normal in G.
(12) Suppose that H and K both have finite index in G. Prove that $[G : H \cap K] \leq [G : H][G : K]$.