MTHSC 851 (Abstract Algebra)
Dr. Matthew Macauley
HW 5
Due Monday Feb. 23, 2009

(1) (a) Let $|G|$ be finite and $N \lhd G$. If $xN \in G/N$ has order a power of p, show that there exists $y \in G$ such that $|y|$ is a power of p and $yN = xN$.
(b) If G/N is abelian and P is a p-Sylow subgroup of G, prove that PN/N is the unique p-Sylow subgroup of G/N.

(2) Let $H_i \lhd G_i$. Prove or give a counterexample.
(a) $G_1 \cong G_2$, $H_1 \cong H_2 \Rightarrow G_1/H_1 \cong G_2/H_2$.
(b) $G_1 \cong G_2$, $G_1/H_1 \cong G_2/H_2 \Rightarrow H_1 \cong H_2$.
(c) $H_1 \cong H_2$, $G_1/H_1 \cong G_2/H_2 \Rightarrow G_1 \cong G_2$.

(3) Give an example of groups H_i, K_j such that $H_1 \times H_2 \cong K_1 \times K_2$ and no H_i is isomorphic to any K_j.

(4) Let G be the additive group $\mathbb{Q}$ of rational numbers. Show that G is not the internal direct product of two of its proper subgroups.

(5) If G is the internal direct product of subgroups G_1 and G_2, show that $G/G_1 \cong G_2$ and $G/G_2 \cong G_1$.

(6) (a) Show that $Z(\prod_\alpha G_\alpha) = \prod_\alpha Z(G_\alpha)$.
(b) Show that $(G_1 \times G_2 \times \cdots \times G_n)' = G_1' \times G_2' \times \cdots \times G_n'$.
(c) Under what circumstances is $G_1 \times G_2 \times \cdots \times G_n$ solvable?

(7) If $f\colon A \to B$ is an equivalence in a category $\mathfrak{C}$ and $g\colon B \to A$ is a morphism such that $gf = 1_A$ and $fg = 1_B$, show that g is unique.

(8) (a) Prove that any two universal (initial) objects in a category $\mathfrak{C}$ are equivalent.
(b) Prove that any two couniversal (terminal) objects in a category $\mathfrak{C}$ are equivalent.

(9) In the category of abelian groups, show that the group $A_1 \times A_2$ together with the homomorphisms

$$\iota_1\colon A_1 \to A_1 \times A_2, \quad \iota_1(x) = (x, e)$$
$$\iota_2\colon A_2 \to A_1 \times A_2, \quad \iota_2(x) = (e, x)$$

is a coproduct for $\{A_1, A_2\}$. Why is this not a coproduct in the category of groups?