MTHSC 851 (Abstract Algebra)
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HW 6
Due Tuesday March 3nd, 2009

(1) (a) Prove that $\sum_{i\in I} A_i$ is a coproduct in the category of abelian groups. Specifically, let $\{A_i \mid i \in I\}$ be a family of abelian groups, and let ι_i be the canonical injections, for $i \in I$. If B is an abelian group and $\{f_i \colon A_i \to B \mid i \in I\}$ a family of homomorphisms, prove there is a unique homomorphism $f\colon \sum_{i\in I} A_i \to B$ such that $f\iota_i = f_i$ for all $i \in I$, and this determines $\sum_{i\in I} A_i$ uniquely up to isomorphism.

(b) Give an example of how the direct product $\prod_{i\in I} A_i$ fails to be a coproduct in the category of abelian groups.

(2) Prove that the free product $\prod\limits_{i\in I}{}^{*} G_i$ is a coproduct in the category of groups.

(3) Let A_1, A_2, A be objects in a category $\mathfrak{C}$, and let $f_i \in \mathrm{Hom}(A, A_i)$ for $i = 1, 2$. Suppose that

$$\begin{array}{ccc} B & \xleftarrow{g_1} & A_1 \\ {\scriptstyle g_2}\uparrow & & \uparrow{\scriptstyle f_1} \\ A_2 & \xleftarrow[f_2]{} & A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} B' & \xleftarrow{g_1'} & A_1 \\ {\scriptstyle g_2'}\uparrow & & \uparrow{\scriptstyle f_1} \\ A_2 & \xleftarrow[f_2]{} & A \end{array}$$

are pushouts for (A, A_1, A_2, f_1, f_2). Prove that B and B' are equivalent.

(4) Give an example of a group that is solvable but not nilpotent.

(5) Show that subgroups and homomorphic images of nilpotent groups are nilpotent.

(6) A pair of homomorphisms $K \xrightarrow{f} G \xrightarrow{g} H$ is said to be *exact* at G if $\mathrm{Im}(f) = \ker g$. A sequence $1 \to K \xrightarrow{f} G \xrightarrow{g} H \to 1$ is called a *short exact sequence* if it is exact at each of K, G, and H.

(a) Show that if $K \lhd G$, $f\colon K \to G$ is the inclusion map and $g\colon G \to G/K$ is the canonical quotient map, then $1 \to K \xrightarrow{f} G \xrightarrow{g} G/K \to 1$ is a short exact sequence.

(b) Show that $1 \to K \xrightarrow{f} G \xrightarrow{g} H \to 1$ is short exact if and only if f is 1–1, g is onto, and $\mathrm{Im}(f) = \ker g$. Conclude that then K is isomorphic with a normal subgroup of G and that $G/f(K) \cong H$.

(c) Suppose $1 \to K \to G \to H \to 1$ is a short exact sequence. Show that G is solvable if and only if both K and H are solvable.

(d) Give an example of a short exact sequence $1 \to K \to G \to H \to 1$ for which K and H are nilpotent but G is not.

(7) If G is a group and $x \in G$ define the *inner automorphism* f_x by setting $f_x(y) = xyx^{-1}$, for all $y \in G$. Write $I(G)$ for the set of all inner automorphisms of G.

(a) Show that $I(G) \leq \mathrm{Aut}(G)$.

(b) Show that $I(G) \cong G/Z(G)$.

(c) If $I(G)$ is abelian show that $G' \leq Z(G)$. Conclude that G is nilpotent.

(d) Compute $\mathrm{Aut}(S_3)$.

(8) Let G be a finite group in which every maximal subgroup is normal.

(a) Prove that G is nilpotent. [*Hint*: If not, then take a non-normal Sylow subgroup $P \leq G$, and choose a maximal $M \leq G$ containing $N_G(P)$. Now, take $x \in G \setminus M$ and look at xPx^{-1}.]

(b) Show that every maximal subgroup of G has prime index.

(9) Let N be a nontrivial normal subgroup of a nilpotent group G. Prove that $N \cap Z(G) \neq 1$.