MTHSC 851 (Abstract Algebra)
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HW 7
Due Thursday March 26th, 2009

(1) Verify that if $S = \{1\}$ and $\mathbb{N} = \{1, 2, 3, \dots\}$, then $(\mathbb{N}, +)$ is a free semigroup on S, with $j(1) = 1$.
(2) Prove that if a free semigroup X exists on S, it is unique up to isomorphism.
(3) If $S = \{a\}$ let $F = \{a^n : n \in \mathbb{Z}\}$ (take $a^0 = 1$), and define $a^m \cdot a^n = a^{m+n}$ and $\phi(a) = a = a^1$. Show that F is a free group on S. Thus a free group on a single "generator" a can be taken to be the infinite cyclic group $\langle a \rangle$.
(4) (a) Let S be a set. The group with presentation (S, R) where $R = \{[s, t] \mid s, t \in S\}$ is called the *free abelian* group on S – denote it by $A(S)$. Prove that $A(S)$ has the following universal property: if G is any abelian group and $\varphi \colon S \to G$ is any set map, then there is a unique group homomorphism $f \colon A(S) \to G$ such that $f|_S = \varphi$.
(b) Deduce that if A is a free abelian group on a set of cardinality n, then
$$A \cong \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \qquad (n \text{ factors}).$$
(5) Show that every nonidentity element in a free group F has infinite order.
(6) Let F be a free group and let N be the subgroup generated by the set $\{x^n \mid x \in F,\ n \text{ a fixed integer}\}$. Show that $N \lhd F$.
(7) Let (F, ϕ) be a free group on a set S.
(a) If $|S| \geq 2$, show that F is not abelian.
(b) For any $T \subseteq S$, show that there exists a normal subgroup $N \lhd F$ such that $(F/N, \pi\phi|_T)$ is a free group on T.
(c) Show that $(\langle \phi(T) \rangle, \phi|_T)$ is a free group on T.
(8) For a positive integer n, let $\mathfrak{C}_{\leq n}$ be the category of nilpotent groups of class at most n. Prove or disprove that free objects always exist in $\mathfrak{C}_{\leq n}$.
(9) Let F be a free object on S, and F' a free object on S' in a concrete category $\mathfrak{C}$. Prove that F and F' are equivalent.